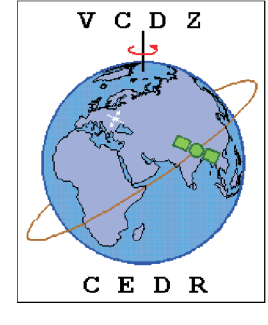


Position of Reflecting Points in Bistatic Satellite Altimetry: Theoretical Solutions for Ellipsoid



S. Olivík¹, M. Kočandrlová², J. Kostecký³, J. Klokočník⁴



¹ Department of Mathematics, Faculty of Civil Engineering Czech Technical University in Prague, Czech Republic olivik@mat.fsv.cvut.cz

² Department of Mathematics, Faculty of Civil Engineering Czech Technical University in Prague, Czech Republic kocandrlova@mat.fsv.cvut.cz

³ Faculty of Civil Engineering Czech Technical University in Prague, Czech Republic kost@fsv.cvut.cz

⁴ CEDR - Research Institute of Geodesy, Topography and Cartography CZ- 250 66, Zábřeh 98, Czech Republic kost@fsv.cvut.cz

⁴ CEDR - Astronomical Institute of the Academy of Sciences of the Czech Republic CZ- 251 65, Ondřejov Observatory, Czech Republic jkloko@asu.cas.cz

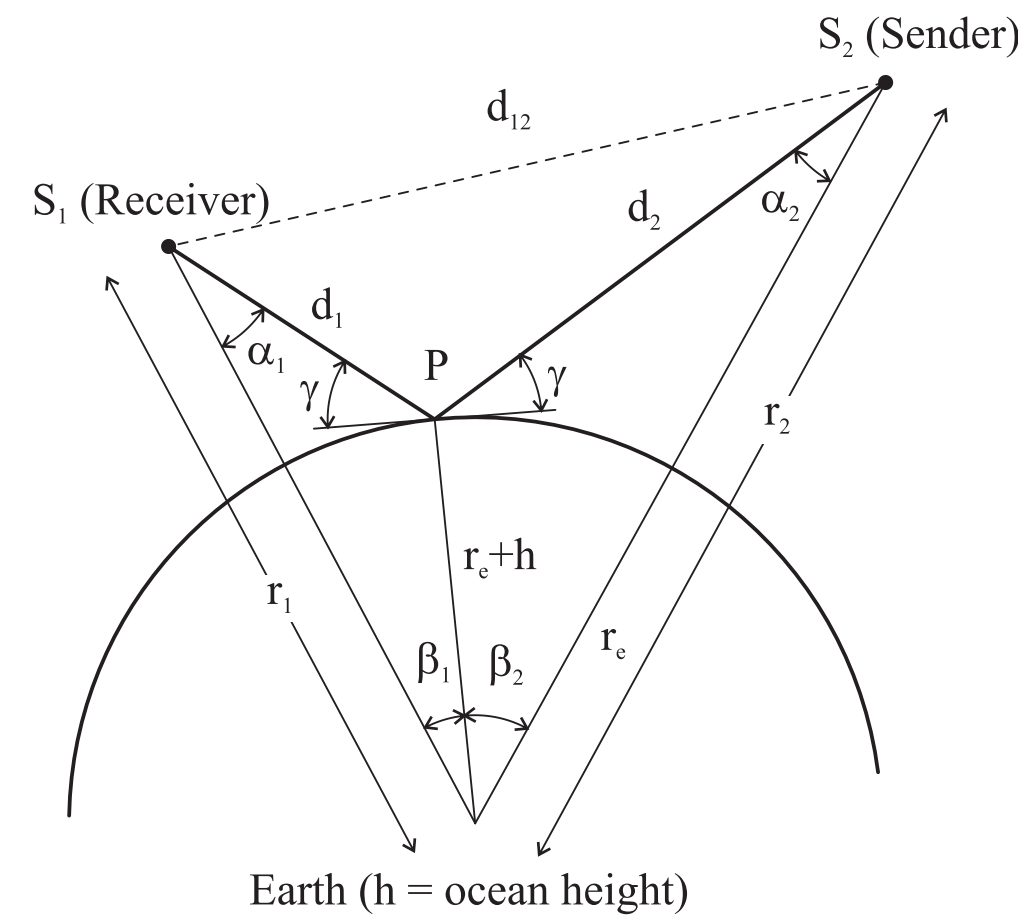
Abstract: This paper deals with satellite bistatic altimetry and namely with computation of the position of the theoretical specular reflecting point on the sea surface. While geometry and accuracy studies, like (Wagner and Klokočník, 2003) may use a spherical Earth as a good first approximation, the actual precise

computation of the position of the reflecting points for scientific use requires a relationship to geoid and sea surface topography; we present here an attempt to solve this problem for a geocentric reference ellipsoid. Three different tasks and their solutions are presented.

Depending on different input data sets, we have made different geometric solutions. If we have full input data set, it leads to the intersection of three quadrics. Partial input data sets may lead to computing minimal distance of two ellipsoids or ellipsoid homothetic to the reference ellipsoid.

Concept of Bistatic Altimetry

Basic idea of Bistatic Altimetry is to receive a radar signal sent by GPS satellites and glanced off the Earth surface. The picture aside shows the situation. If we know the length of the reflected signal, we may be able to compute the point on Earth surface, where the signal glanced off. The three next mathematical models show the way to compute it.

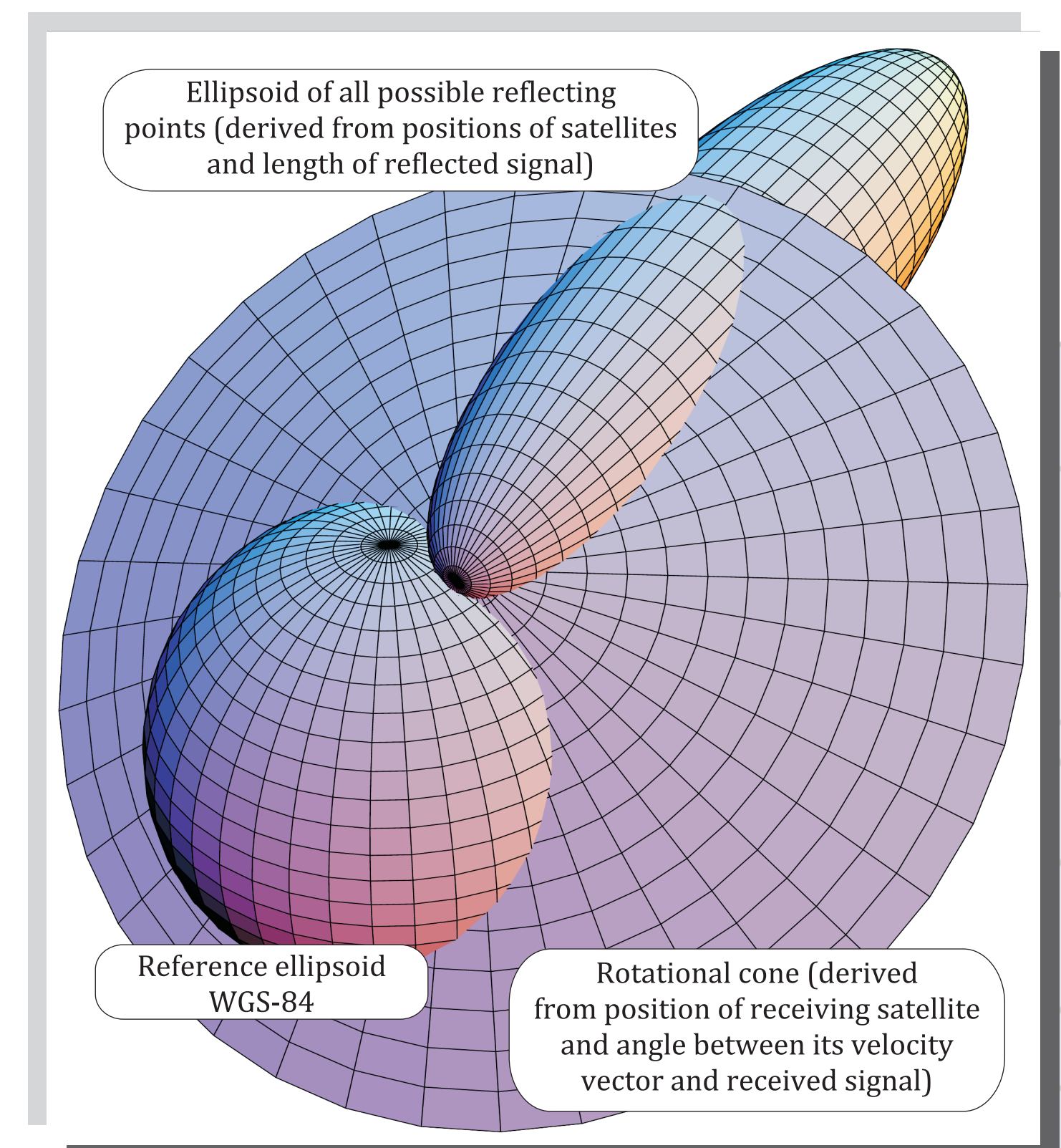


Notation

- S_1 - receiving satellite
- S_2 - sending satellite
- Q_0 - reference ellipsoid WGS-84
- Q_1 - rotational ellipsoid of all possible reflecting points (derived from positions of satellites and length of reflected signal)
- Q_2 - rotational cone (derived from position of receiving satellite and angle between its velocity vector and received signal direction)
- \mathbf{u} - velocity vector of receiving satellite
- θ - angle between the direction of received signal and vector \mathbf{u}

Numerical example - input data

- Positions of satellites (in 10^3 [km])
- $S_1 = [1.70427088; 1.03776088; -6.53202978]$
- $S_2 = [13.43872208; 7.20112522; -21.77247243]$
- length of signal (in 10^3 [km])
- $d_{12} = 21.068077730$
- semi-axis of the reference ellipsoid (in 10^3 [km])
- $a = 6.378137, b = 6.356752$
- additional data for the first model
- $\mathbf{u} = (7.32877; 0.73153; 2.02837)$ [km/sec]
- $\theta = 69.33$ [degrees]



The task is to find the intersection of the quadric Q_1 with the quadric Q_2 and a common point of this intersection and the quadric Q_0 . This common point is the reflecting point we are searching for. The intersecting curve of quadrics Q_1 and Q_2 consists of two ellipses in two planes (ρ_1 and ρ_2). If these planes exist, they must be from a beam of quadrics determined by quadrics Q_1 and Q_2 . The equation of this beam of quadrics is a linear combination of the quadratic equations describing quadrics Q_1 and Q_2 . There are singular quadrics of the beam. The singular quadrics are planes, hence the quadratic equations of each of them is decomposed into two linear equations. We will find these planes and will choose one of them to search for intersection point of all three quadrics. We denote by ρ this plane.

The cone Q_2 and the ellipsoid Q_1 intersect at an intersecting ellipse Q'_1 , lying in the plane ρ . A common point of the ellipse Q'_1 and the reference ellipsoid Q_0 is the desired reflecting point P. Therefore, the point P is a common point of the ellipse Q'_1 and the ellipse Q'_0 , which is the intersection of the ellipsoid Q_0 with the plane ρ .

Singular conic sections of a conic section beam defined by the ellipses Q'_1 and Q'_0 in the plane ρ pass through the point P. The singular conic section of the beam (except Q'_1) has equation $f_1(x) + \lambda_0 g_2(x) = 0$.

There can be two, one or none common points of the ellipses Q'_1 and Q'_0 . This means that there can be two, one or none reflecting point P.

Numerical example

- As a product of this algorithm we obtained two reflecting points
- $P_1 = [1.74884445; 1.07053312; -6.01929799] \cdot 10^3$ [km]
- $P_2 = [1.75458237; 1.00638518; -6.02862494] \cdot 10^3$ [km]
- The position error of points P_1 and P_2 on the reference ellipsoid is $r_1 = 3.6 \cdot 10^{-12}$ [km], $r_2 = 1.8 \cdot 10^{-12}$ [km].

The first model INTERSECTION OF THREE QUADRICS

We know:

- parameters of reference ellipsoid
- positions of both satellites
- length of reflected signal
- velocity vector of receiving satellite
- angle between its velocity vector and received signal direction

We compute an intersection point of three quadrics in the special position

- two rotational ellipsoids
 - oblate rotational reference ellipsoid
 - prolate rotational ellipsoid of all possible reflecting points (derived from positions of satellites and length of reflected signal)
- one rotational cone
 - rotational cone derived from position of receiving satellite and angle between its velocity vector and received signal direction

The second model MINIMUM DISTANCE OF TWO ELLIPSOIDS

We know:

- parameters of reference ellipsoid
- positions of both satellites
- length of reflected signal

We compute minimal distance between two ellipsoids

- oblate rotational reference ellipsoid
- prolate rotational ellipsoid of all possible reflecting points (derived from positions of satellites and length of reflected signal)

The two ellipsoids have one generic point as a tackpoint. We are looking for the points $X_0 \in Q_0$ and $X_1 \in Q_1$ such that

$$\|X_0 - X_1\| = \inf\{\|X_0 - X_1\|; X_0 \in Q_0, X_1 \in Q_1\}$$

When the ellipsoids Q_0 and Q_1 touch each other, the point $X_0^* = X_1^*$ is evidently the required reflective point P.

We evaluate points X_0^* and X_1^* by sequence of successive approximation:

1. *Zero approximation:* Let O_0 and O_1 be central points of ellipsoids Q_0 and Q_1 . Let X_0 be intersection point of ray $O_0 O_1$ with ellipsoid Q_1 . We take pair (X_0, X_1) as the initial approximation.

2. *Determination of next approximations:* In point X_0 we determine unit normal vector \mathbf{n}_0 of ellipsoid Q_0 oriented inward in point X_0 . If a dimension of space

$$V = \langle X_0 - X_1, \mathbf{n}_0, \mathbf{n}_1 \rangle$$

is one, than the pair (X_0, X_1) is a solution.

If a dimension of space V is greater than one, we will choose normal section of ellipsoid Q_0 in point X_0 . We determine its plane ρ that contains point X_1 of the second ellipsoid or contains vector \mathbf{n}_1 . As a plane ρ we can take one plane given by equations

$$[X - X_0, \mathbf{n}_0, X_1 - X_0] = 0, [X - X_0, \mathbf{n}_0, \mathbf{n}_1] = 0,$$

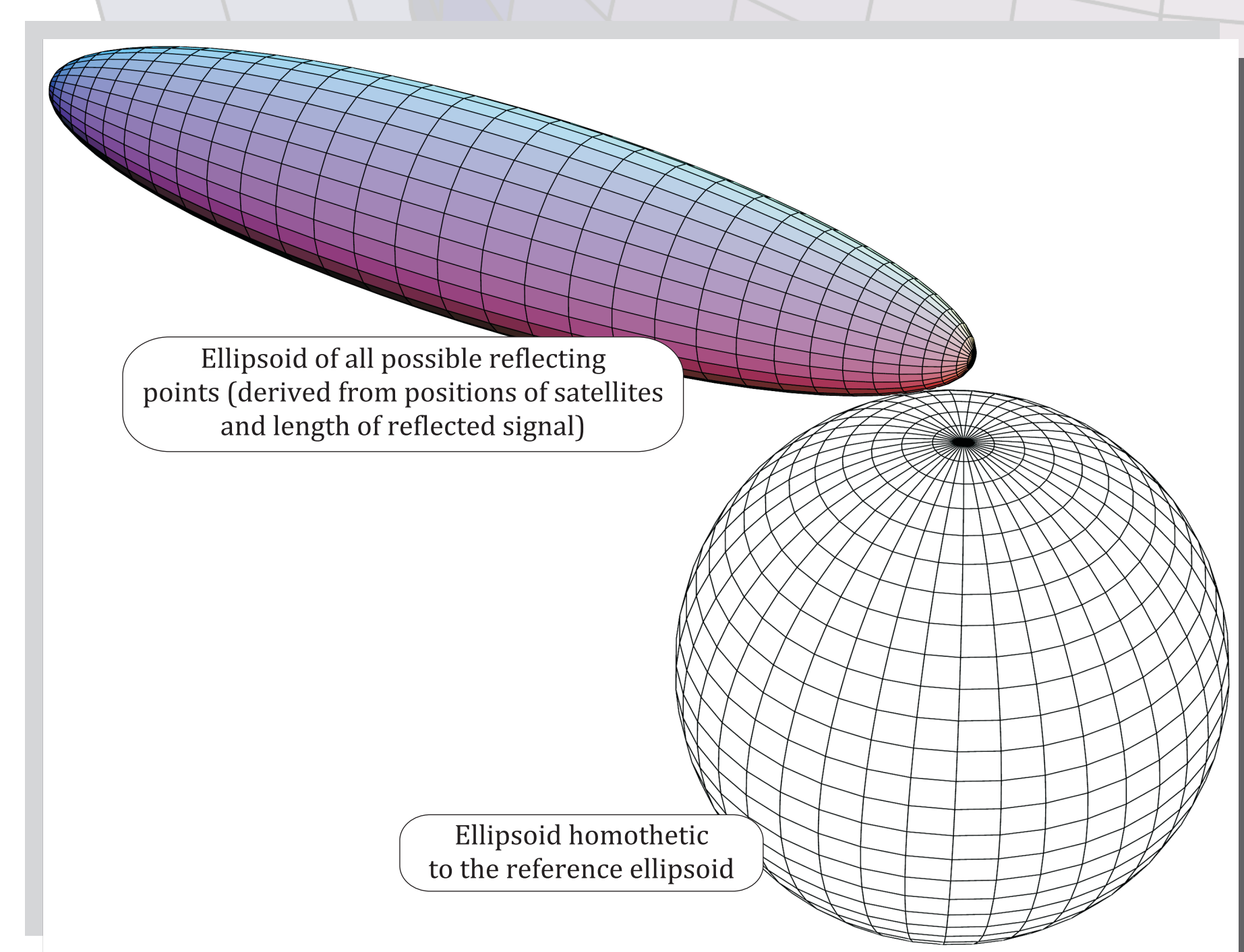
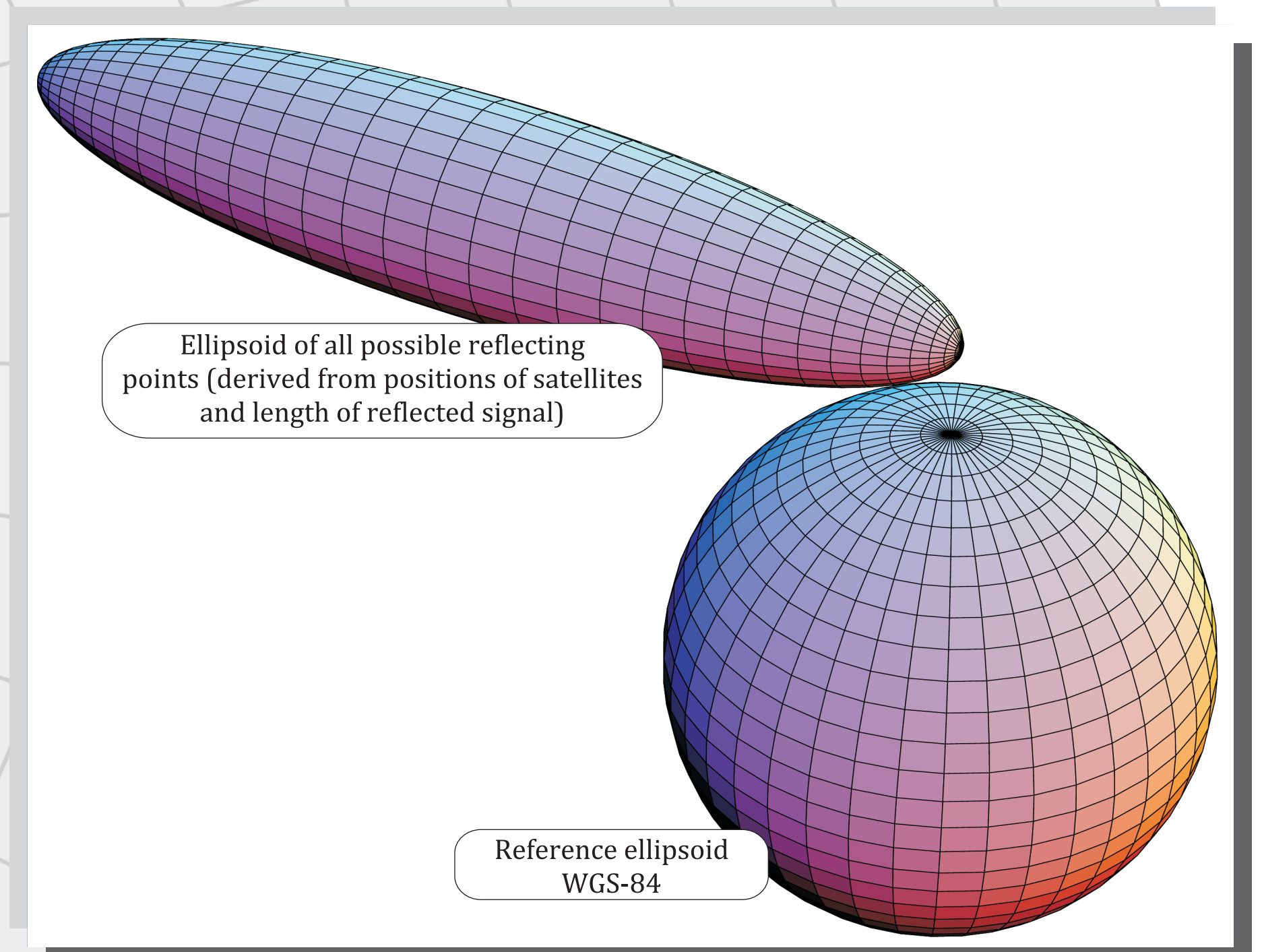
or plane fitting linear combination of these equations and its own equation. Now we determine center of curvature of selected normal section of ellipsoid Q_0 in point X_0 . We mark it as O_0 .

We carry out similarly construction for ellipsoid Q_1 . We receive center of curvature of selected normal section of this ellipsoid. We mark it as O_1 .

We verify (see property of algorithm) whether straight line $O_0 O_1$ intersect both ellipsoids. In positive case we continue according to procedure for determining the next approximation of the points X_0, X_1 .

Numerical example

- After 20 iteration steps we obtain two points that fulfill conditions: 1. $|PP_1|$ is minimum, 2. the normal vectors of both ellipsoids in these points are collinear.
- $P_1 = [1.73699157; 1.03708373; -6.02851686] \cdot 10^3$ [km], $P_2 = [1.73677995; 1.03695738; -6.02777743] \cdot 10^3$ [km]



On ellipsoid of reflecting points Q_1 we are looking for the point, where the homothetic ellipsoid to the reference ellipsoid Q_0 is touching the ellipsoid Q_1 . We know the parameters of reference ellipsoid Q_0 , foci S_1, S_2 and size $2a$ of the major axis of ellipse, which rotation forms the ellipsoid Q_1 .

In affinity, which transforms the reference ellipsoid Q_0 to the sphere, we determine an image of the ellipsoid Q_1 . If q is a half axis of the reference ellipsoid Q_0 , then it holds for this affinity that

$$[x, y, z] \rightarrow [x, y, z/q].$$

This affinity transforms ellipsoid Q_1 to non-rotational ellipsoid Q'_1 . If $f_{ij}, i, j = 0, \dots, 3$ are elements of the matrix \mathbf{F} , than $f'_{ij} = f_{ij}$ for $i, j = 0, 1, 2$, $f'_{3i} = f'_{3i} = qf_{3i}$, $i = 0, 1, 2$, $f'_{33} = q^2 f_{33}$ are the elements of matrix \mathbf{F}' . This transforms the task to the problem how to find a point on the ellipsoid Q'_1 that is the nearest to the reference ellipsoid, also to the adequate sphere. We solve this problem with method of successive approximation.

We set the midpoint of the reference ellipsoid Q_0 , resp. ellipsoid Q'_1 as O_0 , resp. O'_1 , traced point as P' and normal vector of ellipsoid Q'_1 in points P' as \mathbf{n} . Then we repeat the consequent procedure in relation to variable point O .

The initial position of point O is $O = O'_1$.

repeat

- 1) $P = OO_0 \cap Q'_1$
- 2) \mathbf{n} normal vector of Q'_1 in point P' oriented inward
- 3) \mathbf{t} tangent vector of Q'_1 in point P' lying in normal plane in point P' inclusive point O_0

The third model ELLIPSOID HOMOTHETIC TO THE REFERENCE ELLIPSOID

We know:

- parameters of reference ellipsoid
- positions of both satellites
- length of reflected signal

We compute ellipsoid homothetic to the reference ellipsoid

4) r' radius of normal curvature of Q'_1 in point P' and direction of \mathbf{t}

5) $O = P' + r'\mathbf{n}$ center of normal curvature in given direction until (condition)

We can select the condition of the cycle by a conventional line - as a count of iteration, as a distance between position of two consequent iterations of point P' , etc. (10 iterations are sufficient).

To point P' we designate point P on the ellipsoid Q_1 , and intersection point P_0 of ray $O_0 P$ and the reference ellipsoid Q_0 in an inverse affinity. We designate the coefficient of homothetic of midpoint O_0 of the elected reference ellipsoid Q_0 and the traced reference ellipsoid from the pair of corresponding points P, P_0 .

References

- Wagner CA, Klokočník J (2003) The value of ocean reflections of GPS signals to enhance satellite altimetry: data distribution and error analysis. J Geod 77: 128-138
- Martin-Neira M (1993) A passive reflectometry system: application to ocean altimetry. ESA Journal 17: 331-356
- Kostecký J, Klokočník J, Wagner CA (2005) Geometry and accuracy of reflecting points in bistatic satellite altimetry. J Geod 79: 421-430

Acknowledgements

This research has been supported by the grants of Ministry of Education of the Czech Republic, (CEDR) LC 506, FRVŠ G1 392/2005, and by the grant 3407 of the Grant Agency of the Academy of Sciences of the Czech Republic (GA AV ČR).