Chapter 5

Gravitoacoustic waves

5.1 Sound waves in the atmosphere

In this Section, we shall study local behaviour of radial waves, i.e. of small periodic perturbations of hydrostatic equilibrium of a plane-parallel atmosphere. The condition of hydrostatic equilibrium can be obtained from the Navier – Stokes equation (4.6) for a unit volume of fluid in a scalar (gravitational) potential Φ,

$$\rho \frac{dv}{dt} = -\rho \nabla \Phi - \nabla P \ , \tag{5.1}$$

if the the velocity v is set equal to zero. The equation of hydrostatic equilibrium thus reads

$$\rho \nabla \Phi + \nabla P = 0 \ . \tag{5.2}$$

Assuming the atmosphere to be an ideal gas, for which the equation of state (4.7) can be written as

$$P = \frac{k}{m} T \rho \ , \tag{5.3}$$

where m is the mean mass of a free particle (atom, ion, electron), we arrive at a differential equation for the density ρ, if the potential Φ and temperature T are known functions of the radial coordinate r. In particular, for an isothermal atmosphere, $T = constant$, in homogeneous gravity field, $\Phi = gr$, where the gravity acceleration $g = constant$, this equation reads

$$\frac{1}{\rho} \nabla \rho = -\frac{mg}{kT} \ , \tag{5.4}$$

and its solution is the barometric formula

$$\rho = \rho_0 \exp(-\frac{mg}{kT} r) = \rho_0 \exp(-\frac{r}{H}) \ , \tag{5.5}$$

where $H \equiv \frac{kT}{mg}$ is the characteristic height scale of the atmosphere.

Let us now investigate a (dynamic) perturbation of the atmosphere, in which a gas element changes its coordinates

$$x \to x' = x + \delta x \equiv x + \xi(x,t) \ , \tag{5.6}$$

with velocity

$$v = \dot{\xi} \equiv \frac{\partial}{\partial t} \xi(x,t) \ , \tag{5.7}$$

and simultaneously, due to its expansion or compression, it changes also its density and pressure[1]

$$\begin{aligned} \rho &\rightarrow \rho' = \rho + \delta\rho \ , && (5.8) \\ P &\rightarrow P' = P + \delta P \ . && (5.9) \end{aligned}$$

The Lagrangian perturbation of the Lagrangian equation of continuity (4.5)

$$\frac{d\rho}{dt} + \rho(\nabla v) = 0 \tag{5.10}$$

reads

$$\frac{d\delta\rho}{dt} + \rho(\nabla\dot{\xi}) = 0 \ , \tag{5.11}$$

and hence, integrating with time, the perturbation of density is determined by the divergence of the displacement[2]

$$\delta\rho = -\rho(\nabla\xi) \ . \tag{5.12}$$

For fast perturbations, we can assume that the pressure varies according to the adiabatic law

$$P \sim \rho^{\gamma} \ , \tag{5.13}$$

where $\gamma = \frac{5}{3}$ for an ideal gas. Consequently, the perturbation of the pressure is related to the perturbation of the density by

$$\delta P = \gamma \frac{P}{\rho} \delta\rho \ , \tag{5.14}$$

hence, substituting (5.12), we find

$$\delta P = -\gamma P(\nabla\xi) \ . \tag{5.15}$$

The Lagrangian perturbation of the Navier – Stokes equation (5.1) reads

$$\rho\ddot{\xi} = -g\delta\rho - \delta\nabla P \ . \tag{5.16}$$

Substituting here from (5.12), (5.1), (5.15) and the barometric formula (5.5) for P, we find

$$\begin{aligned} \rho\ddot{\xi} &= g\rho(\nabla\xi) - \delta_E \nabla P - \xi\nabla^2 P = -(\nabla\xi)\nabla P - \nabla\delta_E P - \xi\nabla^2 P = \\ &= -\nabla(\delta_E P + \xi\nabla P) = -\nabla(\delta P) = \gamma\nabla(P\nabla\xi) = \\ &= \gamma P(\nabla^2\xi - \frac{1}{H}\nabla\xi) \ . \end{aligned} \tag{5.17}$$

This is the wave-equation for ξ,

$$\ddot{\xi} = c^2(\nabla^2\xi - \frac{1}{H}\nabla\xi) \ , \tag{5.18}$$

[1] The perturbation δ of any physical quantity f is here the Lagrangian perturbation (i.e. that percieved by the element of fluid), $\delta f \equiv f'(x+\xi) - f(x)$. There can also be introduced an Eulerian perturbation (i.e. that measured at a fixed point x), $\delta_E f \equiv f'(x) - f(x)$. Obviously, $\delta f = f'(x) + \xi\nabla f - f(x) = \delta_E f + \xi\nabla f$ (apart from terms of higher orders of the perturbation) and only the Eulerian perturbation commutes with the partial time or space derivatives. Both perturbations are identical for quantities constant in the unperturbed state (in particular for ξ and v, which are zero in the unperturbed atmospheres and hence $\delta\xi \equiv \xi$ and $\delta v \equiv v$). Note that in literature (e.g. [7]) the Eulerian perturbation is often denoted by δ and the Lagrangian perturbation by Δ, which is consistent with their mathematical equivalence with the synchronous and asynchronous variation, resp. However, [9] and their followers use prime (') for the Eulerian and δ for the Lagrangian perturbation.

[2] Alternatively, we can start from an Eulerian perturbation of the Eulerian form of the equation of continuity, $\frac{\partial\rho}{\partial t} + \nabla(\rho v) = 0$, and find the solution $\delta_E\rho = -\nabla(\rho\xi)$. Note that Eqs. (7.3) and (7.4) of [5] are thus incorrect.

where the speed of sound is

$$c = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma k T}{m}} \; . \tag{5.19}$$

Unlike the case of a homogeneous gas, the second term on the right-hand side of Eq. (5.18) causes a change of the amplitude of ξ. Because the density of energy (e.g. kinetic energy) of the wave is proportional to $\rho\dot{\xi}^2$, let us rescale the amplitude to another variable $\xi \to X = \sqrt{\rho}\xi$, the square of which will be directly proportional to the energy density independently of the local value of the density. Substituting this new variable into Eq. (5.18), we find

$$\begin{aligned} \ddot{X} &= c^2\sqrt{\rho}(\nabla - \frac{1}{H})\nabla(X\rho^{-1/2}) = c^2\sqrt{\rho}(\nabla - \frac{1}{H})(\nabla X\rho^{-1/2} + \frac{1}{2H}X\rho^{-1/2}) = \\ &= c^2\sqrt{\rho}(\nabla^2 X\rho^{-1/2} + \frac{1}{H}\nabla X\rho^{-1/2} + \frac{1}{4H^2}X\rho^{-1/2} - \frac{1}{H}\nabla X\rho^{-1/2} - \frac{1}{2H^2}X\rho^{-1/2}) = \\ &= c^2(\nabla^2 X - \frac{1}{4H^2}X) \; . \end{aligned} \tag{5.20}$$

Using the Fourier transform

$$X(x,t) = A(k,\omega)\mathrm{e}^{i(\omega t - kx)} \; , \tag{5.21}$$

Eq. (5.20) turns into an algebraic equation for the amplitude A

$$(\omega^2 - c^2k^2 - \frac{c^2}{4H^2})A(k,\omega) = 0 \; . \tag{5.22}$$

This linear homogeneous equation has a non-trivial solution for A, if the frequency satisfies the dispersion relation

$$\omega = \pm\sqrt{c^2k^2 + \omega_c^2} \; , \tag{5.23}$$

where the cut-off frequency

$$\omega_c = \frac{c}{2H} = \frac{g}{2}\sqrt{\frac{\gamma m}{kT}} \; . \tag{5.24}$$

5.2 Gravity waves

In the previous Section, we assumed that the whole horizontal plane in the atmosphere oscillates simultaneously up and down, i.e. the wave-vector has a horizontal component equal to zero. Here, we shall study the opposite limit, in which only individual blobs of gas with small horizontal size will oscilate up and down in their environment, which is in the mean in the hydrostatic equilibrium. The blob can thus expand horizontally to hold its inner pressure in equlibrium with the unperturbed atmosphere. It means that the Eulerian perturbation of pressure is zero and the Lagrangian perturbation reads

$$\delta P = \xi\nabla P \; . \tag{5.25}$$

We assume that the expansion of the blob is again adiabatic, i.e. Eq. (5.14) is still valid and it determines the variation of density[3]

$$\delta\rho = \frac{\rho}{\gamma P}\xi\nabla P \; . \tag{5.26}$$

[3]Note that Eq. (5.12) is still valid for the vector ξ, and it determines its total divergence both in vertical and horizontal direction.

Generally, the total gravity and buoyancy force acting on the blob of mass M reads

$$f = M(-\frac{1}{\rho}\nabla P - \nabla\Phi) \ , \tag{5.27}$$

and it is equal to zero in the equilibrium. The perturbation of this force caused by the displacement of the blob[4] reads

$$\delta f = M(\frac{\delta\rho}{\rho^2}\nabla P - \frac{1}{\rho}\delta\nabla P) \ . \tag{5.28}$$

Consequently, the blob will move with acceleration

$$\ddot{\xi} = \frac{\delta f}{M} = \frac{1}{\rho}(\frac{\delta\rho}{\rho}\nabla P - \delta\nabla P) = \frac{1}{\rho}(-g\delta\rho - \delta_E\nabla P - (\xi\nabla)\nabla P) = -\frac{g}{\rho}(\delta\rho - (\xi\nabla)\rho) \ , \tag{5.29}$$

where we have used twice Eq. (5.2) and the fact that $\delta_E\nabla P = \nabla\delta_E P = 0$. Substituting here from Eq. (5.26), we arrive at the equation of motion for ξ

$$\ddot{\xi} = -\frac{g}{\rho}(\frac{\rho}{\gamma P}\nabla P - \nabla\rho)\xi \ , \tag{5.30}$$

which is the standard equation of a harmonic oscillator with the circular frequency (so called Brunt - Väisälä frequency) N, which can be expressed (using Eq. (5.3)) as

$$N^2 = g(\frac{1}{\gamma P}\nabla P - \frac{1}{\rho}\nabla\rho) = -g(\frac{\gamma - 1}{\gamma P}\nabla P - \frac{1}{T}\nabla T) \ . \tag{5.31}$$

A negative value of N^2 (i.e. an imaginary value of N) in a case of large (negative) gradient of T would indicate a convection instability. In the case of the isothermal atmosphere

$$N = \sqrt{\frac{\gamma - 1}{\gamma}\frac{g}{H}} = g\sqrt{\frac{\gamma - 1}{\gamma}\frac{m}{kT}} = \frac{2\sqrt{\gamma - 1}}{\gamma}\omega_c \simeq 0.979796\omega_c \tag{5.32}$$

for an ideal gas.

In the case of a general wave in the atmosphere, both effects (i.e. the dilution of density and the buyoancy) influence its propagation depending on the horizontal k_h and the radial (vertical) k_r component of the wave-vector. The dispersion relation for these general gravito-acoustic waves then reads

$$(\omega^2 - \omega_c^2)\omega^2 - (k_h^2 + k_r^2)c^2\omega^2 + k_h^2 c^2 N^2 = 0 \ . \tag{5.33}$$

For $k_h = 0$ this equation reduces to Eq. (5.23). For a general value of $k_h = 0$, there are two regions of ω, for which a non-negative value of k_r^2 may exist. One is the region of high-frequency pressure waves, and the other is of low-frequency gravity waves.

Exercise 7 *Find the commutation relations between the Lagrangian perturbation and partial derivatives (with respect to time or coordinate) for a general quantity $f(x)$. Prove that the Lagrangian perturbation commutes with the Lagrangian time derivative $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (v\nabla)$. Cf. [7], Chapter 6.2.*

[4] In a planeparallel atmosphere $g = \nabla\Phi = constant$, so that its variation is zero. However, for a perturbation of a spherical star the corresponding additional term may appear.