Chapter 4

Model atmospheres and their applications

This chapter contains some selected topics from chapters 7 to 15 of [3] and some additional comments and examples.

4.1 Hydrostatic equilibrium and hydrodynamics

Classical plane-parallel model atmospheres are assumed to be in hydrostatic equilibrium. This assumption is inconsistent for spherical atmospheres. Some observations also confirm that the atmospheres of real stars often continue by a less or more dense outflow of gas called a stellar wind. The hydrostatic equilibrium is a limiting case of the hydrodynamics which has many applications in other fields of astrophysics and physics as well as its technical applications. We will thus derive first the equations of hydrodynamics (starting from a phenomenological insight as well as from the kinetic description of the gas), next we will simplify them for the hydrostatic limit, and finally we will apply them for a spherical stellar wind.

Phenomenological hydrodynamics. The hydrodynamic description of liquids and gasses is usually based on the phenomenological view of a continuum, the physical characteristics of which (like the density ρ, the velocity $\vec{v}$ etc.) are some functions of time t and spatial position described by a chosen 'Eulerian' coordinates x (i.e., $f = f(t, x)$). The evolution of the continuum is described by a partial differential equations for these quantities. It is possible to identify individual parts of the continuum and to follow their motion $x = x(t, x_0)$, where x_0 are the 'co-moving' 'Lagrangian' coordinates. The velocity field is given by

$$v = \frac{\partial}{\partial t} x(t, x_0) \;, \tag{4.1}$$

and the time-derivatives of all quantities can be rewritten from the Eulerian to the Lagrangian form introducing the 'flow-' time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (v\nabla) \;. \tag{4.2}$$

Choosing some region Ω_0 of Lagrangian coordinates we can find its volume V at the time t as

$$V(t) = \int_\Omega d^3x = \int_{\Omega_0} \left| \frac{\partial x(t, x_0)}{\partial x_0} \right| d^3x_0 \ . \tag{4.3}$$

Its flow- time derivative reads

$$\frac{d}{dt}V(t) = \int_{\Omega_0} \frac{\partial}{\partial t} \left| \frac{\partial x(t, x_0)}{\partial x_0} \right| d^3x_0 = \int_{\Omega_0} \sum_i \frac{\partial v^i}{\partial x^i} \left| \frac{\partial x(t, x_0)}{\partial x_0} \right| d^3x_0 = \int_\Omega (\nabla v) d^3x \simeq V.(\nabla v) \tag{4.4}$$

(where we have used the relation $\frac{\partial}{\partial t}\frac{\partial x^i}{\partial x_0^j} = \frac{\partial v^i}{\partial x^k}\frac{\partial x^k}{\partial x_0^j}$ in each term of the derivated determinant; the last equality is valid for an infinitesimally small region Ω).

Let us suppose now, that the total mass $M = \rho V$ of the region is conserved. We thus arrive at the equation of continuity which reads

$$\frac{d\rho}{dt} + \rho(\nabla v) = \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \ , \tag{4.5}$$

in the Lagrangian or the Eulerian form, resp.

Similarly, if we suppose that the time- derivative of the momentum of contained in the region is given by the total force acting on the element of continuum (i.e., the pressure acting on its surface as well as the density of force acting in its volume), we arrive at the 'Navier – Stokes equation' of motion

$$\rho \frac{dv}{dt} \equiv \rho \left(\frac{\partial v}{\partial t} + (v\nabla)v \right) = f - \nabla P \ , \tag{4.6}$$

where f is the volume density of the force (e.g., $-\rho\nabla\Phi$ in the case of a scalar potential) and P is the pressure (which can be replaced by an anisotropic stress-tensor for a viscose continuum). The value of the pressure is usually assumed to be determined by an equation of state (e.g., for an ideal gas)

$$P = P(\rho, T) \sim nkT \ . \tag{4.7}$$

The temperature T, which corresponds to the density of the inner energy (e.g. $\varepsilon \sim \frac{3}{2}nkT$ for the ideal gas), is either given by the thermodynamic equilibrium with some other physical system (e.g., with the radiation field), or it must be solved from the equation of continuity of the energy. Constructing this equation, it must be taken into account that the change of the inner energy of the element of gas is influenced (in addition to other processes with the efficiency $\dot{E}$) also by the work exerted by the change of the volume against its pressure, $\frac{d}{dt}(\varepsilon V) = \dot{E} - P\frac{dV}{dt}$. For instance, in the case of an ideal gas, this equation reads

$$\frac{d}{dt}\varepsilon + \frac{5}{3}\varepsilon(\nabla v) = \dot{\varepsilon} \ , \tag{4.8}$$

where on the right-hand side is the volume-density of the other processes like the efficiency of the thermonuclear reactions, the energy absorbed from the radiation (which can be found from the solution of the equation of radiative transfer), or the divergency of the conductive flow of thermal energy.

Exercise 5 *(1) Prove by an explicit calculation in two dimensions that*

$$\frac{\partial}{\partial t} \left| \frac{\partial(x^1, x^2)}{\partial(x_0^1, x_0^2)} \right| = \left(\frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} \right) \left| \frac{\partial(x^1, x^2)}{\partial(x_0^1, x_0^2)} \right| \ .$$

(2) Prove the same relation splitting the transform $x_0 \to x(t)$ into two steps $x_0 \to x(t') \to x(t)$ and fixing $t' = t$. (3) Generalize the proof to higher dimensions.

Hydrodynamic approximation from moments of Boltzmann equation. If we write down Boltzmann equations (2.7) for all kinds of particles of the gas (these particles are generally non-relativistic – unlike the photons treated in Section 2.1) and we calculate their moments in the momentum space, we arrive at partial differential equations for the moments of the distribution functions f. The first three moments are just the quantities: numerical density of particles

$$n = \frac{1}{m}\rho = \int f d^3p \,, \tag{4.9}$$

density of their momentum

$$\pi^i = \rho v^i = \int p^i f d^3p \,, \tag{4.10}$$

and the stress tensor

$$mT^{ij} = m(\rho v^i v^j + \tau^{ij}) = \int p^i p^j f d^3p \tag{4.11}$$

(τ is the proper stress tensor with respect to the comoving frame, while T corresponds to the coordinate frame), i.e., the quantities we want to solve from the equations of hydrodynamics. Let us show in the simple case of particles moving in a scalar potential field $\Phi(x)$ that the moment equations are equivalent to the above given equations of continuity, motion etc.

The Hamiltonian of one particle reads

$$H = \frac{p^2}{2m} + m\Phi(x) \,, \tag{4.12}$$

and the Hamiltonian equations of motion

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} \,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} = -m\frac{\partial \Phi}{\partial x^i} \,, \tag{4.13}$$

obviously satisfy the Liouville theorem ($\frac{\partial \dot{x}^i}{\partial x^i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0$). The Boltzmann equation can thus be written in the explicit form

$$\frac{\partial f}{\partial t} + \frac{p_i}{m}\frac{\partial f}{\partial x^i} - m\frac{\partial \Phi}{\partial x^i}\frac{\partial f}{\partial p_i} = \left(\frac{\delta f}{\delta t}\right)_{\rm c} \,. \tag{4.14}$$

The zero-order moment (i.e., $\int d^3p$) of this equation is

$$\frac{\partial n}{\partial t} + \frac{1}{m}\frac{\partial \pi^i}{\partial x^i} = \left(\frac{\delta n}{\delta t}\right)_{\rm c} \,, \tag{4.15}$$

where on the right-hand side is the moment of the collisional term, which gives the number density of particles of the particular type created by the collisions (chemical reactions). The integral of the third term on the left-hand side is zero because it is an integral of the derivative of f, which must vanish at infinite p. Substituting here n and pi from Eqs. (4.9) and (4.10), we arrive, after multiplication by m and summation for all kinds of particles, at the equation of continuity (4.5).

The first moments of Eq. (4.14) can be obtained multiplying it by p^j and integrating it in p,

$$\frac{\partial \pi^j}{\partial t} + \frac{\partial T^{ji}}{\partial x^i} + \rho\frac{\partial \Phi}{\partial x^j} = \left(\frac{\delta \pi^j}{\delta t}\right)_{\rm c} \,, \tag{4.16}$$

where we have used the integration 'per partes' $\int p^j \frac{\partial f}{\partial p_i} = -\delta^j_i \int f$. The collisional term on the right-hand side gives the momentum transferred to the particles in collission. It must vanish for a closed system of particles, but it can give also the force exerted by the radiation pressure, if the radiative processes are included. Using Eq. (4.15), a more common form of the equation of motion of a one-component gas can be obtained (cf. Eq. (4.6))

$$\rho\left(\frac{\partial v^j}{\partial t}+v^i\frac{\partial v^j}{\partial x^i}\right)+\frac{\partial \tau^{ji}}{\partial x^i}+\rho\frac{\partial \Phi}{\partial x^j}=\left(\frac{\delta \pi^j}{\delta t}\right)_{\mathrm{c}}-v^j\left(\frac{\delta \rho}{\delta t}\right)_{\mathrm{c}} . \tag{4.17}$$

In a similar way, the second moments of Eq. (4.14) can be obtained (integrating $\int d^3pp^jp^k/m$)

$$\frac{\partial T^{jk}}{\partial t}+\frac{\partial Q^{ijk}}{\partial x^i}+\pi_j\frac{\partial \Phi}{\partial x^k}+\pi_k\frac{\partial \Phi}{\partial x^j}=\left(\frac{\delta T^{jk}}{\delta t}\right)_{\mathrm{c}} , \tag{4.18}$$

where on the right-hand side there are contributions of the collisional term to the stress tensor, and Q^{ijk} is the moment of the third order.

In a close correspondence with the moment equations of the radiative transfer, each moment of the Boltzmann equation for the particles of gas contains higher order moments of the distribution function. To get a closed set of equations, it is necessary to estimate them as some functions of lower-order moments (as it is done for moments of radiation field by means of the Eddington factor). This is a matter of different versions of the hydrodynamic approximation. One of the simplest possibilities is to assume the relaxation processes (caused by the collisions between the particles of gas) so effective, that the distribution function f is the equilibrium Maxwellian distribution

$$f=n(2\pi mkT)^{-\frac{3}{2}}\exp\left(-\frac{(p-mv)^2}{2mkT}\right) , \tag{4.19}$$

where the height and the position of the maximum of the Gaussian curve are according to (4.9) and (4.10) given by ρ and v, which must satisfy Eqs. (4.15) and (4.17). Following Eq. (4.11), the stress tensor for the equilibrium distribution (4.19) in the comoving system of the gas and it is proportional to the temperature T of the gas,

$$\tau^{ij}=\delta^{ij}P=\delta^{ij}nkT=\delta^{ij}\frac{2\varepsilon}{3} , \tag{4.20}$$

where P is the pressure and ε is the density of thermal energy. The temperature can be either fixed (e.g., due to the equilibrium with the radiation), or it (or ε) can be found as a solution of the equation of continuity of energy,

$$\left(\frac{\partial}{\partial t}+v^i\frac{\partial}{\partial x^i}\right)\varepsilon+\frac{5}{3}\varepsilon\frac{\partial v^i}{\partial x^i}=\left(\frac{\delta \varepsilon}{\delta t}\right)_{\mathrm{c}} , \tag{4.21}$$

which can be obtained from the trace (i.e., $\sum_{j=k}$) of Eq. (4.18), if Q and T are expressed from (4.19) in terms of n, v, T. The second term on the left-hand side gives the above mentioned change of the inner energy due to the work exerted by the change of volume against the pressure of the gas. The collisional term on the right-hand side is either zero for an adiabatic motion of one-component gas, or it can give the exchange of the energy with the other components of gas or the radiation (the later being dominant in the case of the radiative equilibrium). The other components of Eq. (4.18) as well as the higher order moments of the equation yield the criteria of reasonability of the assumption (4.19). If the collisions are not enough efficient to damp the perturbations of the equilibrium distribution arising due to the diffusion of particles in the non-homogeneous or differenetially moving gas, the non-diagonal components of the stress-tensor, the heat conduction etc. must be also solved from the higher moment equations including the appropriate expressions

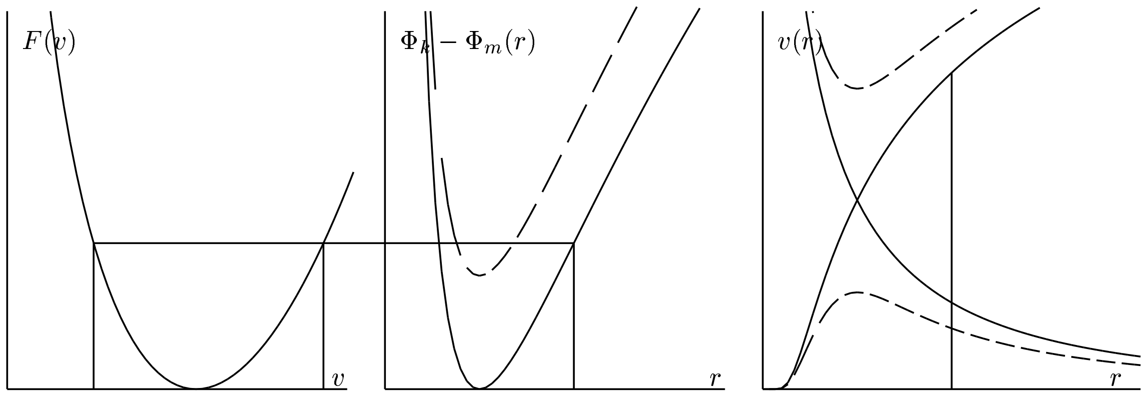

Figure 4.1: Spherically symmetric flow of gas.

for the collisional term. A simple semi-empirical model of the collisional terms can be obtained if the assumption of the exact Maxwellian distribution (4.19) is replaced by the 'BGK'- model (Bhatnagar, Gross, Krook)

$$\left(\frac{\delta f}{\delta t}\right)_{\mathrm{c}} = -\frac{f - f_0}{\Delta t} , \tag{4.22}$$

where f_0 is the equilibrium distribution with the values of n, v (and facultatively also T) corresponding to the the local distribution f and Δt is a relaxation time. Because the moments of this collisional term are proportional to the difference between the actual moments of f and the moments of f_0, the former can be expressed as a sum of the later plus a perturbation proportional to the left-hand side of the moment equation. Being the coefficient of the proportionality, the relaxation time Δt thus gives the viscosity, thermal conductivity etc. The main weak point of the approximation (4.22) is that in reality the relaxation is not uniform in the p- space and consequently it can not to fit all these coefficients simultaneously.

Exercise 6 *Derive the moment equations for a gas of charged particles in external electromagnetic field.*

Stellar wind. Let us apply the hydrodynamic equations to a stationary spherically symmetric outflow of gas, which is a model of the so called stellar wind. If we suppose the velocity field to be $v^i = v(r)r^i/r$, we can find the Eulerian equation of continuity (4.5) in the form

$$0 = \nabla^i \left(\frac{\rho(r)v(r)}{r} r^i\right) = r^{-2}\frac{d}{dr}\left(\rho(r)v(r)r^2\right) , \tag{4.23}$$

which can be integrated as

$$\rho(r)v(r)r^2 = h , \tag{4.24}$$

where the constant $h = -\dot{M}/4\pi$ is the mass flowing per unit time into a unit solid angle (it is positive for the wind outflow or negative in the case of spherically symmetric accretion). The vector equation of motion (4.6) or (4.17) has the only non-zero component in the radial direction

$$\rho(r)\frac{dv}{dr} = -\rho g - \frac{d}{dr}(P + P_r) , \tag{4.25}$$

where $g = \frac{d}{dr}\Phi$ is the gravity acceleration and P_r is the radial component of the stress tensor of radiation field (2.6), the gradient of which can be – using the moment (2.28) of the equation of radiative transfer – expressed also by means of the radial flow of the radiation.

According to the equation of state (4.7), the pressure P of the gas is linked to the density ρ and temperature T. If the behavior of the temperature is known, e.g., if it is fixed by the interaction with the radiation field as nearly isothermic, or if it is an adiabatic cooling etc., then the pressure P is determined by the density ρ, which can be, according to (4.24), expressed as a function of v, for which we thus arrived at an ordinary differential equation

$$\left(v - \frac{kT}{m}\frac{1}{v}\right)\frac{dv}{dr} = -g - r^2\frac{d}{dr}\left(\frac{kT}{mr^2}\right) + \frac{\bar{\kappa}}{c}F_r \ . \tag{4.26}$$

By its integration we find an implicit algebraic equation for v in the form

$$F(v) = \Phi_k - \Phi_m(r) \ , \tag{4.27}$$

where the function $F(v)$ on the left-hand side reads, e.g., in the isothermic case,

$$F(v) = \frac{1}{2}(v^2 - v_k^2) - \frac{kT}{m}\ln\left|\frac{v}{v_k}\right| \tag{4.28}$$

and it has a minimum equal to zero at a critical point

$$v_k = \sqrt{\frac{kT}{m}} \ . \tag{4.29}$$

On the right-hand side of Eq. (4.27) there is a modified potential $\Phi_m(r)$, which has in the isotermic case (and for a constant effective opacity per unit mass $\bar{\kappa} = \bar{\alpha}/\rho = \int \alpha F_r d\nu / (\rho \int F_r d\nu)$) the form

$$\Phi_m(r) = \int \left(g - 2\frac{kT}{mr} - \frac{\bar{\kappa}}{c}F_r\right) dr = -\frac{GM}{r} - 2\frac{kT}{m}\ln(r) + \frac{\bar{\kappa}L}{4\pi cr} \ , \tag{4.30}$$

where G is the gravity constant, M the mass of the star and $L = 4\pi r^2 F_r$ its luminosity. This potential also has a minimum in a critical point

$$r_k = \frac{m}{2kT}\left(GM - \frac{\bar{\kappa}L}{4\pi c}\right) \ . \tag{4.31}$$

To ensure the existence of a smooth solution of $v(r)$ for all r, the integration constants must satisfy the inequality $\Phi_k \geq \Phi_m(r_k)$. If $\Phi_k > \Phi_m(r_k)$, there exist two solutions, one with a subcritical velocity and the other with supercritical velocity (cf. the dashed lines in Fig. 4.1). In the case $\Phi_k = \Phi_m(r_k)$ these solutions are joined in the critical point, and they can be combined into a solution corresponding either to the stellar wind, which is asymptotically static for $r \to 0$ and dynamical for $r \to \infty$, or to the radial accretion, for which it is vice versa and the velocity is negative (cf. full lines in Fig. 4.1). There must be satisfied the condition that the luminosity of the star is below the Eddington limit, i.e. $L < \frac{4\pi c}{\bar{\kappa}}GM$), and hence the attractive gravity potential prevails above the repulsive pressure of the radiation.