Chapter 2

Radiative transfer

This chapter corresponds to chapters 1, 2, 3 and 6 of [3].

2.1 Equation of radiative transfer

In classical approximation (i.e., not quantum-field view), the radiation field can be described either phenomenologically using the specific intensity, or in the kinetic approximation as a gas of photons, or as a superposition of electromagnetic waves.[1] Unlike the Mihalas' Eq. (1-2), it is more advantageous in the kinetic description to use the distribution function f normalized per unit quantum state[2] $\frac{d^3x d^3p}{h^3}$, where the momentum of photon

$$\vec{p} = \frac{h\nu}{c}\vec{n} \ , \tag{2.1}$$

and hence

$$d^3p = \frac{h^3}{c^3}\nu^2 d\nu d^2\omega \ , \qquad d^3x = cdtd^2S \ . \tag{2.2}$$

The specific intensity I is thus related to f as

$$I = \frac{2h\nu^3}{c^2} f \ , \tag{2.3}$$

where the factor 2 corresponds to possible polarization states of photons. Polarization is usually neglected in models of stellar atmospheres, despite it is known that it can cause an error of several percents of the intensity even in planeparallel atmosphere dominated by Thomson scattering. A

[1] Cf. [3, pp. 7-9] – The choice of units (SI, CGSE or CGSM) influences the coefficients in equations of electromagnetic field and must be carefully watched if the effects of electromagnetic field on the radiating plasma are investigated (e.g., the magnetohydrodynamics of the atmosphere or the Zeeman effect in line profiles). However, the present aim is simply to demonstrate the consistency of the description of radiative field and its stress-energy tensor in both the particle- and field- approach.

[2] The main advantage of the normalization per unit element of phase-space volume consists in the Lorentz-invariance of the phase-space element and thus also of f. It can be proved in relativistic kinetic theory, however, intuitively it is obvious from the Lorentz contraction of the space-volume, which is inversely proportional to relativistic transformation of momentum. Unlike f, the specific intensity I, which includes the energy of photons and which is normalized per element of spherical coordinates $d^2\omega d\nu$, is thus influenced by Doppler-shift and aberration in treatment of radiative transfer in differentially moving media. The choice of quantum state as the unit of phase-space makes simpler the expressions for probabilities of quantum transitions and also the quantum statistics; the Planck's black-body radiation is described by the simple Bose-Einstein distribution $f = (\exp(\frac{h\nu}{kT}) - 1)^{-1}$.

more sophisticated formulation of radiative transfer can be done in terms of Stoke's parameters describing the partial linear as well as circular polarization of the light.

If the distribution function f of photons or the specific intensity I are known, the basic properties of the radiation field can be calculated. The most important are the first three moments of the distribution, which are the components of the stress-energy tensor, i.e., the density of the energy of radiation

$$\varepsilon_r = 2\int h\nu f \frac{d^3p}{h^3} = \frac{1}{c}\int I_\nu d\nu d^2\Omega \ , \tag{2.4}$$

the density of momentum π_r^i or the density of the flow of radiative energy

$$F_r^i = c^2\pi_r^i = 2\int h\nu n^i c f \frac{d^3p}{h^3} = \int n^i I_\nu d\nu d^2\Omega \ , \tag{2.5}$$

and the stress tensor (pressure) of the radiation

$$T_r^{ij} = 2\int h\nu n^i n^j f \frac{d^3p}{h^3} = \frac{1}{c}\int n^i n^j I_\nu d\nu d^2\Omega \ . \tag{2.6}$$

In the kinetic view of the radiation field, the equation of radiative transfer is the Boltzmann equation for the gas of photons

$$\frac{d}{dt}f(t,x,p) \equiv \frac{\partial}{\partial t}f + \frac{dx^i}{dt}\frac{\partial}{\partial x^i}f + \frac{dp^i}{dt}\frac{\partial}{\partial p^i}f = \left(\frac{\delta f}{\delta t}\right)_c \ , \tag{2.7}$$

which is, generally, a consequence of the Liouville theorem, i.e., the conservation of phase-space volume. This is valid for any Hamiltonian motion. The phase-space density f of particles is then constant for collision-free particles, or its change is proportional to the collisional term $(\frac{\delta f}{\delta t})_c$ on the right-hand side of the Boltzmann equation (2.7). This term gives the number of particles which will appear in the unit phasse-space volume per unit time due to the collisions (minus the number of particles which will disappear). Changing the parametrization to the length $s = ct$ measured along the ray and substituting for f from Eq. (2.3) in terms of the specific intensity I, the equation of radiative transfer reads[3]

$$\frac{d}{ds}I(t,x,n,\nu) = \frac{\partial}{c\partial t}I + \frac{dx^i}{ds}\frac{\partial}{\partial x^i}I + \frac{dn^i}{ds}\frac{\partial}{\partial n^i}I + \frac{d\nu}{ds}\nu^3\frac{\partial}{\partial\nu}\nu^{-3}I = \left(\frac{\delta I}{\delta s}\right)_c \ . \tag{2.8}$$

In the case of photons in Cartesian coordinates in a non-refractive medium, the momentum (i.e., both the direction $\vec{n}$ and the frequency ν) is constant and $\frac{dx^i}{ds} = n^i$. Usually we deal with a time-independent problem only, so that the term with the time-derivative also vanishes and the equation of radiative transfer thus reduces to its standard form

$$\frac{d}{ds}I(t,x,n,\nu) = n^i\frac{\partial}{\partial x^i}I = \left(\frac{\delta I}{\delta s}\right)_c \ . \tag{2.9}$$

However, in curvilinear coordinates (e.g., spherical) or in the case of refraction or gravitational bending of light, the change of n^i must also be included. In the case of the change of ν, e.g., due to the Doppler shift in a differentially moving medium or due to the gravitational redshift, the term with $\frac{\partial}{\partial\nu}$ must also be included (but it can not be commuted with the multiplicator ν^3 from Eq. (2.3)).

[3]Note, that only two components of the unit vector $\vec{n}$ are algebraically independent. This is why the angles of spherical coordinates in the p-space are often used instead of the components.

The collisional term on the right-hand side of Eq. (2.8) gives now the amount of energy radiated into the beam of light per its unit length (into its unit cross-section in unit solid angle and unit frequency interval). We can suppose that it is a sum of two terms

$$\left(\frac{\delta I}{\delta s}\right)_c = \epsilon - \alpha I \ , \tag{2.10}$$

the emission (ϵ) and the absorption[4] ($-\alpha I$). The later is negative and proportional to the intensity of light. The absorption coefficient (or opacity) α is thus the probability that a photon from the beam will be absorbed in a unit length and it is proportional to the volume density of the absorbers multiplied by their effective cross-section. We usually replace the emission coefficient (emisivity) ϵ by the source function S defined by the relation

$$\epsilon = \alpha S \ . \tag{2.11}$$

The equation of radiative transfer thus reads

$$\frac{d}{ds} I(t, x, n, \nu) = \alpha(S - I) \ . \tag{2.12}$$

There exists also the stimulated (induced) emission, which is positive and proportional to the intensity. For the purpose of solution of the equation of radiative transfer (2.12), we use to include it into the linear absorption term as a negative absorption. As it will be explained in Chapter 3, the stimulated emission is a consequence of the boson nature of photons, and it thus takes place in all radiative processes including both the true absorption/emission as well as all kinds of scattering.[5]

2.2 Formal solution of radiative transfer

The equation of radiative transfer along the ray (in terms of the optical depth τ, where $d\tau = \alpha ds$),

$$\frac{d}{d\tau} I = S - I \ , \tag{2.13}$$

has the formal solution

$$I(\tau) = \int_{-\infty}^{\tau} S(t) \mathrm{e}^{-(\tau - t)} dt \ , \tag{2.14}$$

which can be interpreted as the superposition of exponentially damped contributions S of light radiated at depths $t < \tau$ of the ray (τ is increasing in direction of the ray in this case).

For the radiation field in a plane-parallel geometry the individual rays are parametrized by direction cosine μ and the formal solution of the equation of radiative transfer (with τ measured in the direction $\mu = -1$)

$$\mu \frac{\partial}{\partial \tau} I = I - S \tag{2.15}$$

thus reads

$$I(\tau, \mu) = \int_{\tau}^{\tau_b(\mu)} S(t, \mu) \mathrm{e}^{-\frac{t-\tau}{\mu}} d\frac{t}{\mu} \ , \tag{2.16}$$

[4]Note that [3] uses the letters η instead of ϵ and χ instead of α.

[5]The notice in [3, p. 24] about its existence for a definit upper state only is thus wrong. In the opposite case it would lead to violation of the 3[rd] law of thermodynamics.

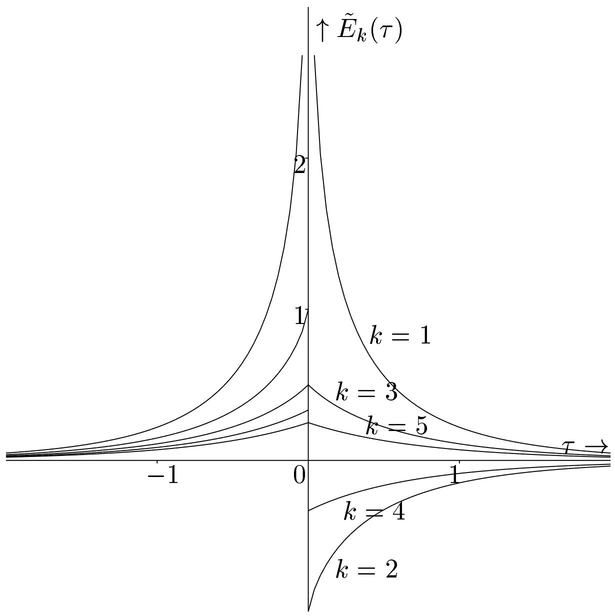

Figure 2.1: Modified exponential integral.

where the boundary optical depth $\tau_b(\mu) = \infty$ for outgoing rays, $\mu > 0$, and $\tau_b(\mu) = 0$ for in-going rays, $\mu < 0$ (cf. [3, Eqs. (2-50,51)]). The source function S is usually supposed to be isotropic ($\frac{\partial S}{\partial \mu} = 0$). However, it need not be the case, e.g., if there is an anisotropic scattering. Let us thus suppose that it can be expressed in the form of power-law series, $S = \sum_{l=0}^{\infty} S_l \mu^l$. The formal solutions (cf. [3, Eqs. (2-57,59,60)]) for moments $J \equiv M_0$, $H \equiv M_1$, $K \equiv M_2$ (i.e., up to the coefficients 4π or $4\pi c$ the components of the stress-energy tensor – cf. Eqs. (2.4), (2.5) and (2.6)), and generally any moment can thus be generalized and simplified as

$$M_k(\tau) \equiv \int_{-1}^{1} I(\tau,\mu)\mu^k \frac{d\mu}{2} = \tag{2.17}$$

$$= \sum_l \int_{-1}^{1} \int_{\tau}^{\tau_b(\mu)} S_l(t)\mu^{k+l} \mathrm{e}^{-\frac{t-\tau}{\mu}} d\frac{t}{\mu}\frac{d\mu}{2} =$$

$$= \sum_l \int_{\infty}^{0} S_l(t) \int_0^1 \mu^{k+l-1} \mathrm{sign}^{k+l-1}(t-\tau)\mathrm{e}^{-\frac{|t-\tau|}{\mu}} \frac{d\mu}{2} dt =$$

$$= \sum_l \frac{1}{2}(S_l * \tilde{E}_{k+l+1}) \; , \tag{2.18}$$

if we define $S_l(\tau) = 0$ for $\tau \leq 0$ to hold the convolution $*$ in the variable τ defined in the standard range of integration $(-\infty, \infty)$. Here we have introduced a modified exponential integral

$$\tilde{E}_k(\tau) = \mathrm{sign}^{k-1}(-\tau) E_k(|\tau|) \; , \tag{2.19}$$

which is defined in the whole range of real numbers and which differs from the standard exponential integral

$$E_k(x) = x^{k-1}\Gamma(1-k,x) \tag{2.20}$$
$$= \int_0^1 \mu^{k-2}\exp\left(-\frac{x}{\mu}\right) d\mu \tag{2.21}$$
$$= \int_1^\infty y^{-k}\exp(-xy)\,dy \tag{2.22}$$

(defined for positive numbers) by the sign-convention only. It is obvious that the convolution with functions $\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$ gives the kernels of integral operators Λ, Φ and X, resp. (cf. [3, Eqs. (2-58,61,62)]). The discontinuity of $\tilde{E}_{2k}$ at zero results in δ-function in its derivative,

$$\frac{d}{d\tau}\tilde{E}_k(\tau) = \tilde{E}_{k-1}(\tau) - \frac{1+(-1)^k}{k-1}\delta(\tau) \; . \tag{2.23}$$

Applying this relation between operators to the source function one arrives at an infite set of differential equations for the moments

$$\frac{d}{d\tau}M_k(\tau) = M_{k-1}(\tau) - \sum_l \frac{1-(-1)^{k+l}}{2(k+l)}S_l \; , \tag{2.24}$$

which can also be obtained as moments of the transfer equation (2.15) (i.e., multiplying it by μ^{k-1} and integrating by $d\mu$), and which in the special case of isotropic S reduces Eqs. (2-71) and (2-79) of [3]

$$\frac{d}{d\tau}H = J - S \; , \tag{2.25}$$
$$\frac{d}{d\tau}K = H \; . \tag{2.26}$$

These equations can also be obtained by the integration of Eq. (2.15) with $d\mu$ or $\mu d\mu$, resp. They describe the conservation of radiative energy and momentum, which is important for the vertical structure of the atmosphere. For the total balance of energy and momentum in all frequencies these equations must be expressed in a frequency independent radial coordinate, e.g., z, and integrated in frequencies

$$\frac{d}{dz}\int H d\nu = \int \alpha(S-J)d\nu \; , \tag{2.27}$$
$$\frac{d}{dz}\int K d\nu = -\int \alpha H d\nu \; . \tag{2.28}$$

The first one states that the divergence of the radiative flux is the difference between the density of energy emitted and absorbed. In radiative equilibrium we suppose this difference to be zero. The second one gives the gradient of the pressure of radiation as the density of momentum transferred from the radiative flux to the matter by the absorption.

The formula (2.23) rewritten in the form

$$\tilde{E}_k(\tau) = \frac{d}{d\tau}\tilde{E}_{k+1}(\tau) + \frac{1-(-1)^k}{k}\delta(\tau) \; , \tag{2.29}$$

can be usefull for practical calculations. For instance, in the case of a polynomial source function $S(\tau,\mu) = h(\tau)S_{lm}\mu^l\tau^m$ (h being the Heaviside function), we find[6]

$$\begin{aligned}
M_k &= \frac{1}{2}S_{lm}(h\tau^m) * \tilde{E}_{k+l+1} = \frac{1+(-1)^{k+l}}{2(k+l+1)}S_{lm}h\tau^m + \frac{1}{2}S_{lm}(h\tau^m)' * \tilde{E}_{k+l+2}(\tau) = \\
&= \frac{1+(-1)^{k+l}}{2(k+l+1)}S_{lm}h\tau^m + \frac{1}{2}S_{lm}\delta_0^m\tilde{E}_{k+l+2}(\tau) + \frac{1}{2}S_{lm}(mh\tau^{m-1}) * \tilde{E}_{k+l+2}(\tau) = \\
&= \frac{1}{2}S_{lm}h\left(\frac{1+(-1)^{k+l}}{k+l+1}\tau^m + \frac{1-(-1)^{k+l}}{k+l+2}m\tau^{m-1} + ...\right) \\
&\quad + \frac{1}{2}S_{lm}\left(\delta_0^m\tilde{E}_{k+l+2}(\tau) + m\delta_0^{m-1}\tilde{E}_{k+l+3}(\tau) + ...\right) \ . \qquad (2.30)
\end{aligned}$$

In the special case of isotropic S linear in τ we thus arrive at Eqs. (2-63,64,65) of [3]. Recurrently, one can obtain from Eq. (2.29) also the expression for the modified exponential integrals as series of derivatives of δ-functions[7]

$$\tilde{E}_k(\tau) = \sum_{n=0}^{\infty} \frac{1-(-1)^{k+n}}{k+n}\frac{d^n}{d\tau^n}\delta(\tau) \ . \qquad (2.31)$$

Applying this relation to the source function, we can express all moments as combinations of its derivatives

$$M_k(\tau) = \sum_{nl} \frac{1+(-1)^{k+l+n}}{2(k+l+n+1)}\frac{d^n}{d\tau^n}S_l \ , \qquad (2.32)$$

i.e., in the special case of isotropic S we obtain Eqs. (2-89a,b,c) of [3]. and for the linear S also Eqs. (2-63,64,65) of [3].

Exercise 1 *(a) Find the behavior of $\tilde{E}_k(x)$ at $x \to 0$ and $x \to \pm\infty$, (b) prove Eqs. (2.23) and (2.31), (c) prove the algebraic recurrent formula for $\tilde{E}_k(x)$,*

$$\tilde{E}_{k+1}(x) = \frac{1}{k}\left(\mathrm{sign}^k(-x)\mathrm{e}^{-|x|} + x\tilde{E}_k(x)\right) \ .$$

Exercise 2 *Derive the equation of radiative transfer and the corresponding moment equations for the case of spherical symmetry.*

2.3 Problems of scattering and grey atmosphere

In the formal solutions (2.16) or (2.18) of the transfer equation (2.15) we supposed the source function $S(\tau,\mu)$ to be given (for each frequency) independently on the solution of radiation field. However, the emissivity of a coherent scattering is proportional to the specific intensity of the light,

[6] Here we use the property (A.4) of the convolution and the relation $(h\tau^m)' = \delta_0^m\delta(\tau) + mh\tau^{m-1}$.

[7] The convergent expansion of E_k reads

$$E_k(x) = \frac{(-x)^{k-1}}{(k-1)!}\left(-\ln x - \gamma + \sum_{m=1}^{k-1}\frac{1}{m}\right) - \sum_{m=0,m\neq k-1}^{\infty}\frac{(-x)^m}{(m-k+1)m!} \ ,$$

where the Euler's constant $\gamma \simeq 0.5772156649$.

e.g., for the isotropic scattering $\epsilon_s = \sigma J$. If the emissivity of standard absorption and emission $\epsilon_a = \kappa B$ (cf. [3, Sec. 2-1]), the total source function

$$S = \frac{\kappa B + \sigma J}{\kappa + \sigma} = (1 - \varsigma)B + \varsigma J \ , \tag{2.33}$$

where $\varsigma = \frac{\sigma}{\kappa+\sigma}$ is the so-called single scattering albedo. Eq. (2.15) of radiative transfer is thus differential in τ and integral in μ

$$\mu \frac{\partial}{\partial \tau} I = I - \varsigma \int_{-1}^{1} I \frac{d\mu}{2} - (1 - \varsigma)B \ . \tag{2.34}$$

The formal solution (2.18) for the moments of radiation field, which in the present simplified case of isotropic source function reads for the mean intensity $J \equiv M_0$

$$J = \frac{1}{2}(S * \tilde{E}_1) \equiv \Lambda(S) = \Lambda[(1 - \varsigma)B] + \Lambda[\varsigma J] \ , \tag{2.35}$$

is thus an operator equation[8]

$$J - \Lambda[\varsigma J] = \Lambda[(1 - \varsigma)B] \ , \tag{2.36}$$

for J. This equation can be formally solved as

$$J = [1 - \Lambda\varsigma]^{-1} \Lambda[(1 - \varsigma)B] \ , \tag{2.37}$$

however, to find the inverse operator $[1 - \Lambda\varsigma]^{-1}$ is not, generally, a simple task. In the case of small relative values ς of the scattering, the emissivity of scattering can be treated as a small perturbation, for which the correction can be found iteratively by the so called Λ-iteration

$$J_{(0)} = \Lambda[(1 - \varsigma)B] \ , \qquad J_{(n+1)} = J_{(0)} + \Lambda\varsigma J_{(n)} \ , \tag{2.38}$$

which is equivalent to the expression

$$[1 - \Lambda\varsigma]^{-1} = \sum_{n=0}^{\infty} [\Lambda\varsigma]^n \ , \tag{2.39}$$

for the function of the operator $\Lambda\varsigma$ in terms of the sum of geometric series. Unfortunately, in practice, the value of ς is close to 1, for which values the Λ-iteration converges very slowly.

Approximate Λ-iterations. The problem of convergence of Λ-iterations can be overcome (or at least decreased) in numerical calculations by the following method. If we denote the right-hand side of Eq. (2.36) as R and we split the Λ-operator into two parts

$$\Lambda\varsigma = A + (\Lambda\varsigma - A) \ , \tag{2.40}$$

Eq. (2.36) reads

$$(1 - A)J - (\Lambda\varsigma - A)J = R \ , \tag{2.41}$$

[8] This equation is the Fredholm equation of the second kind. It can be alternatively written for unknown functions S or $J - B$,

$$S - \varsigma\Lambda(S) = (1 - \varsigma)B \ , \qquad [1 - \Lambda\varsigma](J - B) = \Lambda B - B \ .$$

The operator A is chosen to approximate $\Lambda\varsigma$, however, at the same time to be sufficiently simple so that $1 - A$ could be easily inverted. The equation for J can then be rewritten into the form

$$J - (1-A)^{-1}(\Lambda\varsigma - A)J = (1-A)^{-1}R \; , \tag{2.42}$$

which again is formally similar to Eq. (2.36), but the operator $(1-A)^{-1}(\Lambda\varsigma - A)$ is now smaller and the iteration process of the type (2.39) converges much faster and in a wider region of ς. This is why the abbreviation 'ALI' of the method is also interpreted as 'accelerated Lambda-iteration'. The simplest choice of A is $A = \varsigma \times 1$, i.e., the first term in the expression (2.29) for $\tilde{E}_1$, however, the method is much more efficient for a three-diagonal matrix A in the representation in discrete values of optical depth.

Grey atmosphere. In the respect of the above discussed problems with convergence in a scattering atmosphere, the extreme case is that of a grey atmosphere in radiative equilibrium. The assumption of greyness consists in frequency independence of the total (absorption plus scattering) opacity α. The transfer equation (2.15) can thus be integrated in frequency and it has the same form for the bolometric intensity and also its moments (2.27), (2.28) etc. are identical with (2.25), (2.26) resp. Due to the additional assumption of radiative equilibrium we thus have

$$\frac{d}{d\tau}H = J - S = 0 \tag{2.43}$$

for the bolometric quantities, which thus behave as a pure scattering $\varsigma = 1$ (despite the emission need not balance the absorption in individual frequencies). The equation of radiative transfer thus is the linear homogeneous equation

$$\mu\frac{\partial}{\partial\tau}I = I - \int_{-1}^{1} I\frac{d\mu}{2} \; , \tag{2.44}$$

and its (nontrivial) solutions must be parameterized by the required value of the radiative flux (which is constant due to the radiative equilibrium). An approximate or exact solution of this equation can be found by different methods (cf. [3, Sec. 3-3, 3-4]). A simple but useful approximation is the Milne - Eddington approximation, which can be found by the following procedure. According to Eq. (2.26), the radiation pressure is exactly

$$K = H\tau + c \tag{2.45}$$

in the radiative equilibrium. We can define the Eddington factor

$$f \equiv \frac{K}{J} = \frac{\int \mu^2 I d\mu}{\int I d\mu} \; . \tag{2.46}$$

Due to its definition, $f \in (0,1)$; f can be close to 0 if there prevails the radiation in the direction of the plain of symmetry. This could happen in a finite slab (which can model, e.g., an accretion disk). However, in a stellar atmosphere we expect almost isotropic radiation in deep layers, in which case $f = \frac{1}{3}$. Close to the surface the radiation should be more concentrated in directions with higher μ, because the corresponding beams are coming from deeper and hotter regions. f is thus higher than $\frac{1}{3}$ here. It could approach the value of 1 for the outward peaking of radiation $I(\mu) \sim \delta(\mu - 1)$ at large distances of spherical stars, however, for a planeparallel atmosphere only a moderate concentration to $\mu \sim 1$ can be expected. Note, that for a known dependence $f(\tau)$ (and $S(\tau)$), Eqs. (2.25) and (2.26) form a close set of equations for J and H, which is used in some numerical methods, because f is of the order of 1 and it can thus be estimated without a large error. To fit properly the asymptotic behaviour at $\tau \to \infty$, we choose $f = \frac{1}{3}$ in the Milne – Eddington approximation, and we thus have

$$S = J = 3K = 3H(\tau + c) \ , \tag{2.47}$$

where c is any real integration constant. Following the formal solution (2.16),

$$I(\tau, \mu) = 3H(\tau + \mu + c) \tag{2.48}$$

for $\mu > 0$, and for $\mu < 0$ the intensity approaches this behaviour at large τ, where the missing radiation from above the surface is not perceivable. In fact, for any value of c, this is an exact solution of Eq. (2.44), however, it does not satisfy the required boundary conditions at the surface (which corresponds to the wrong values of f close to the surface). To minimize this disadvantage of the solution, we can fix c to the value, for which the source function (2.47) will give the correct value $H(\tau = 0)$ integrated with the proper boundary conditions, i.e., according to (2.18). Owing to the linear dependence of S on τ, we can use Eq. (2.30) and we find $c = \frac{2}{3}$. The exact solution of the grey atmosphere reads

$$S = 3H(\tau + q(\tau)) \ , \tag{2.49}$$

where the Hopf function q increases monotonically from $q(0) \simeq 0.577$ to $q(\infty) \simeq 0.710$, which agrees with the Milne – Eddington approximation quite well, at least in limits of applicability of the grey model to real atmospheres.