Appendix B

Variational principles

B.1 Systems with finite degrees of freedom

In classical mechanics the motion of a system with n degrees of freedom $x^i|_{i=1}^n$ (e.g., a motion of point mass) can be expressed by the variatioal principle

$$0 = \delta S \ , \tag{B.1}$$

where the action S is a function of the motion $x = x(t)$ of the system given by the lagrangian $L = L(t, x, \dot{x})$, i.e.,

$$S = \int L(t, x, \dot{x}) dt \ , \tag{B.2}$$

and $\dot{} \equiv \frac{d}{dt}$. From the principle (B.1) then follow equations of motion in the form of the Euler – Lagrange equations

$$0 = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \ . \tag{B.3}$$

If we define the generalized momentum $p_i = \frac{\partial L}{\partial \dot{x}^i}$, and we solve from here $\dot{x} = \dot{x}(t, x, p)$,[1] then in the phase space $\{x, p\}$ can be written equations equivalent to (B.3) in the form of the Hamilton canonical equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \ , \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \ , \tag{B.4}$$

where the Hamiltonian

$$H(t, x, p) = p_i \dot{x}^i - L(t, x, \dot{x}) \ . \tag{B.5}$$

Equations (B.4) can also be obtained as the Euler – Lagrange equations from the Hamilton – Jacobi variational principle

$$0 = \delta S = \delta \int (p_i \dot{x}^i - H(t, x, p)) dt \ , \tag{B.6}$$

where x and p are taken as independent variables. From the equations (B.4) there follows

$$\frac{d}{dt} H(t, x, p) = \frac{\partial H}{\partial t} \ , \tag{B.7}$$

[1] Equations $p_i = p_i(t, x, \dot{x})$ can be solved with respect to $\dot{x}$, if the matrix $\frac{\partial p_i}{\partial \dot{x}^j} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$ is regular.

hence the Hamiltonian is an invariant of motion just if it is not explicitely dependent on time.

From the point of view of quantum mechanics, the principle of extremal (least) action is valid because in fact the system is spreading out from the initial event $(t_{\rm i}, x_{\rm i})$ and it can get to the end one $(t_{\rm f}, x_{\rm f})$ on all trajectories $x(t)$ with these terminal points, and each of these trajectories contributes a term $\exp(\frac{i}{\hbar}S)$ to the wave function $\psi(t_{\rm f}, x_{\rm f})$. The contributions of trajectories close to the extremal of the action add with the same (or close) phase, while the contributions of more different trajectories mutually cancel – see [1]. If the ending event is changed $(t_{\rm f}, x_{\rm f}) \equiv (t, x) \rightarrow (t + \Delta t, x + \Delta x)$, the action (B.2) varies, i.e.,

$$\Delta S(t,x) = L\Delta t + \frac{\partial L}{\partial \dot{x}^i}\delta x^i + \int \left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right]\delta x^i dt \ , \tag{B.8}$$

where at the final point $\Delta x = \delta x + \dot{x}\Delta t$. The last term in (B.8) is zero along the extremal in consequence of (B.3), hence

$$\Delta S(t,x) = \frac{\partial L}{\partial \dot{x}^i}\Delta x^i - \left(\frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L\right)\Delta t = p_i \Delta x^i - H\Delta t \ . \tag{B.9}$$

In Newtonian mechanics where the universal time t exists, are both the action S and the Lagrangian L scalar quantities with respect to any transformation $x \rightarrow x' = x'(t,x)$. However, in relativity the time coordinate $x^0 \equiv ct$ is equivalent to the space coordinates, and each non-zero variation δx of the worldline has at least from the point of view of some observers a non-zero variation also of the time component. Unlike the action which has a Lorentz invariant physical meaning, the Lagrangian in the expression (B.2) is dependent on the chosen coordinate system. To get to the relativistic formulation of the variational principle, we must make the position of all spacetime coordinates equal using the parametric description of the motion $x^\iota = x^\iota(w)|_{\iota=0}^n$, where x^0 is an additional (the $n+1$-st) up to now nonspecified function of the parameter w. For any parameterization the action is then given by the expression

$$S = \int f(x, x')dw \ , \tag{B.10}$$

where $' \equiv \frac{d}{dw}$ and the parametric Lagrangian

$$f(x,x') = t' L(t, x, \dot{x}) \ , \tag{B.11}$$

are not explicitely dependent on w. Because $\dot{x} = \frac{x'}{t'}$, the generalized momentum canonically conjugated to the space coordinates

$$p_i \equiv \frac{\partial f}{\partial x'^i} = \frac{\partial L}{\partial \dot{x}^i} \tag{B.12}$$

is unchanged. If the time coordinate $t(w)$ is also taken as a dynamical variable, the generalized momentum conjugated to t will be

$$p_0 \equiv \frac{\partial f}{\partial t'} = L - \frac{\partial L}{\partial \dot{x}^i}\frac{x'^i}{t'} = -H(t,x,p) \ . \tag{B.13}$$

The space components of the Euler – Lagrange equations obtained from f are a multiple by t' of the equations (B.3), and their time- component is also their linear combination (the sum of multiples by x'). All $n+1$ Lagrange equations are thus consistent with the original time equations, but there is not a unique solution because they are satisfied by any parameterization. To get a uniqueness it is necessary to add to the variational principle (B.1) for the parametric formulation of the dependence $x(w)$ an additional condition fixing the parameterization. This can have the form of an algebraic equation

$$\omega(x, x') = 0 \ , \tag{B.14}$$

where ω is an arbitrary function. Spacetime variations δx^ι are then no more independent, but they must satisfy the linear equation

$$\delta\omega = \frac{\partial\omega}{\partial x}\delta x + \frac{\partial\omega}{\partial x'}\delta x' = 0 \ , \tag{B.15}$$

so that in (B.1) must be taken the variation of (B.10) with the condition (B.14). The Lagrangian variational principle for a parametric formulation of the motion thus reads

$$0 = \delta S = \delta \int f(x, x') + n(w)\omega(x, x')dw \ , \tag{B.16}$$

where n is the Lagrangian coefficient of the variational problem with a condition.

Similarly, the Hamiltonian formalism for a parametric description of the motion can be obtained from the action (B.10), the system of equations (B.12) and (B.13) for t' and x' does not have a unique solution (the condition from the footnote [1] on p. 50 is not satisfied for f given by (B.11)). The expression $p_\iota x'^\iota - f$ formally produced from f for x^ι analogously to the Hamiltonian (B.5) from L for x^i is identically equal to zero, which corresponds to the fact that the Hamiltonian made in this way, which can not be explicitely dependent on w, should be – in analogy with (B.7) – a trivial integral of motion. In view of (B.13) the integral in (B.6) can be expressed as $\int p_\iota x'^\iota dw$ (the Hamilton function for $x^\iota(w)$ would thus be identically equal to 0). However, p_ι are not independent because they must satisfy the condition (B.13). The Hamilton – Jacobi variational principle for parametric description of the motion must be thus in the form

$$0 = \delta \int (p_\iota x'^\iota - N(w)\mathcal{H}(x, p))dw \ , \tag{B.17}$$

where N again is the Lagrangian coefficient, and the so-called superhamiltonian

$$\mathcal{H}(x, p) = p_0 + H(t, x, p) \tag{B.18}$$

is the expression, which put equal to zero gives the condition (B.13). Taking the variation of (B.17) with respect to p_0 we get

$$0 = x'^0 - N(w) \ , \tag{B.19}$$

from where the physical meaning of $N(w)$ is obvious.

B.2 Fields

For a general spacetime field $\varphi_I = \varphi_I(x)$ (where the index "$_I$" denotes individual components of the field φ), the action S is the integral of the Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi)$, i.e.,

$$S = \int \mathcal{L} d^4x \ , \tag{B.20}$$

hence its variation is

$$\begin{aligned} \delta S &= \int \delta\mathcal{L} d^4x = \int \sum_I \left(\frac{\partial\mathcal{L}}{\partial\varphi_I}\delta\varphi_I + \frac{\partial\mathcal{L}}{\partial\varphi_{I,\kappa}}\delta\varphi_{I,\kappa} \right) d^4x \\ &= \int \sum_I \left(\frac{\partial\mathcal{L}}{\partial\varphi_I} - \frac{\partial}{\partial x^\kappa}\frac{\partial\mathcal{L}}{\partial\varphi_{I,\kappa}} \right) \delta\varphi_I d^4x \ , \end{aligned} \tag{B.21}$$

and the field equations read

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi_I} - \frac{\partial}{\partial x^\kappa}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{I,\kappa}}\right) . \tag{B.22}$$

If the Lagrangian (density) does not depend explicitely on the coordinates, then its spacetime variation is given by the change of the field only

$$\frac{\partial \mathcal{L}}{\partial x^\iota} = \frac{\partial \mathcal{L}}{\partial \varphi}\varphi_{,\iota} + \frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}}\varphi_{,\kappa\iota} = \frac{\partial}{\partial x^\kappa}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}}\varphi_{,\iota}\right) , \tag{B.23}$$

(The field index for which this expression should be summed is omitted for brevity). We can thus define the canonical stress-energy tensor

$$T_\iota{}^\kappa \equiv \frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}}\varphi_{,\iota} - \delta^\kappa_\iota \mathcal{L} + t_\iota{}^{[\kappa\lambda]}{}_{,\lambda} , \tag{B.24}$$

which satisfies the conservation law

$$T^{\iota\kappa}{}_{,\kappa} = 0 . \tag{B.25}$$

t is an arbitrary tensor with the marked antisymmetry due to which it cancels in the conservation law.