Appendix A

Differential geometry

A.1 Tensor algebra

In this chapter will be summarized basic properties of vectors, tensors and other geometrical objects in flat spaces like momentum space or the Minkowskian spacetime in special theory of relativity.

Definition 1 *Let T be an n-dimensional vector space over a field*[1] $\mathcal{R}$*. The* **dual vector space** T^* *is the space of linear mappings*

$$\xi : T \to \mathcal{R} \; .$$

$\{\theta^k|_{k=1}^n \in T^*\} = \{e_k|_{k=1}^n \in T\}^*$ *are mutually* **dual bases** *of T and T^*, if*

$$\theta^i(e_j) = \delta^i_j \; . \tag{A.1}$$

Obviously, T^* is also n-dimensional. It is common to identify $T^{**} \equiv T$ and to write

$$\xi(X) = \langle \xi X \rangle = X^i \xi_i \; ,$$

where $X = X^i e_i \in T$ and $\xi = \xi_i \theta^i \in T^*$ (following the Einstein notation, the summation is over identical upper and lower indices).

Definition 2 *The space T^p_q of* **tensors** *of type (p,q) is the space of multilinear mappings*

$$A : \underbrace{T^* \times ... \times T^*}_{p} \times \underbrace{T \times ... \times T}_{q} \to \mathcal{R} \; .$$

In the set of tensors of all types, there is defined the operation **tensor product**

$$\otimes : T^p_q \times T^r_s \to T^{p+r}_{q+s}$$

in such a way that

$$(A \otimes B)(\underbrace{\xi, ...}_{p}, \underbrace{\eta, ...}_{r}, \underbrace{X, ...}_{q}, \underbrace{Y, ...}_{s}) = A(\underbrace{\xi, ...}_{p}, \underbrace{X, ...}_{q}) B(\underbrace{\eta, ...}_{r}, \underbrace{Y, ...}_{s}) \; , \tag{A.2}$$

[1]A field is a commutative and associative ring with a unit, i.e., a set with the operations addition (with respect to which the set is a group) and multiplication. In general relativity we deal mostly with the field of real numbers, sometimes also complex numbers.

and the mapping **contraction** *in i-th and j-th index*

$$\langle \quad \rangle : T^p_q \to T^{p-1}_{q-1} \ ,$$

such that

$$\langle A\rangle(\underbrace{\xi,...,}_{i-1}\underbrace{\eta,...,}_{p-i}\underbrace{X,...,}_{j-1}\underbrace{Y,...}_{q-j}) = \sum_{k=1}^{n} A(\underbrace{\xi,...,}_{i-1}\theta^k,\underbrace{\eta,...,}_{p-i}\underbrace{X,...,}_{j-1}e_k,\underbrace{Y,...}_{q-j}) \ , \tag{A.3}$$

where $\{e_k|_{k=1}^n\} = \{\theta^k|_{k=1}^n\}^*$ are any dual bases.

Obviously $\dim(T^p_q) = n^{p+q}$. The tensor product forms an algebra in the space $\oplus_{p,q=0}^{\infty} T^p_q$ of all tensors. Every (not necessarily dual) couple of bases in T and T^* induces a basis in T^p_q (formed by tensor products of their elements), in which any tensor $A \in T^p_q$ can be expressed by the components

$$A = A^{i_1,...,i_p}_{j_1,...,j_q} e_{i_1} \otimes ... \otimes e_{i_p} \otimes \theta^{j_1} \otimes ... \otimes \theta^{j_q} \ .$$

It can be verified that the definition of contraction is independent of the choice of the couple of dual bases.

A.2 Exterior algebra

Definition 3 *The spaces of* p-**vectors** $\wedge^p T \subset T^p_0$ *or* p-**forms** $\wedge^p T^* \subset T^0_p$ *($p \leq n$) are the subspaces of tensors of types $(p,0)$ or $(0,q)$ resp., which are antisymmetric in all arguments. The* **exterior product** *is the mapping*

$$\wedge : \wedge^p T \times \wedge^q T \to \wedge^{p+q} T \ ,$$

such that

$$(X \wedge Y)(\xi_1, ..., \xi_{p+q}) = \sum_{\sigma} \frac{(-1)^{|\sigma|}}{p!q!} (X \otimes Y)(\xi_{\sigma(1)}, ..., \xi_{\sigma(p+q)}) \ , \tag{A.4}$$

where $|\sigma|$ is the parity of permutation σ of numbers $1,...,p+q$ (i.e., $+1$ for even or -1 for odd permutation). The **interior product** *of a p-form ξ and q-vector X is the $|p-q|$-form (or -vector) formed by contraction of their tensor product in the first $\min(p,q)$ indices,*

$$\langle \xi X\rangle = \langle \xi \otimes X\rangle \ .$$

Obviously, $\dim(\wedge^p T) = \binom{n}{p}$, $\wedge^0 T = \mathcal{R}$, $\wedge^1 T = T^1_0 = T$, $\wedge^1 T^* = T^0_1 = T^*$. An exterior product is antisymmetric, i.e.,

$$(X \wedge Y) = (-1)^{qp} (Y \wedge X) \ , \tag{A.5}$$

linear and asociative. It thus forms (the so-called Grassman's) algebra in 2^n-dimensional space of direct products $\oplus_{p=0}^{n} (\wedge^p T)$.

Definition 4 *Let us choose a* **unit** n-**form** $\epsilon \in \wedge^n T^*$ *on an n-dimensional space T. The n-form ϵ then defines the* **dual mapping**

$$* : \wedge^p T \leftrightarrow \wedge^{n-p} T^*; ,$$

such that for $A \in \wedge^p T$, $\omega \in \wedge^p T^*$,

$$*A = \frac{1}{p!} \langle \epsilon A\rangle \ , \tag{A.6}$$

$$*\omega = \frac{1}{p!} \langle \omega E\rangle \ , \tag{A.7}$$

where $E \in \wedge^n T$ is the **unit** n-**vector**, *for which*

$$*E = 1 \ . \tag{A.8}$$

The following relation is valid

$$* * \omega = (-1)^{p(n-p)} \omega \ . \tag{A.9}$$

The exterior product of p-vectors could be confused with the vector product of vectors, which is also an antisymmetric bilinear operation (sometimes denoted by the same symbol $\wedge$) but defined only on a 3-dimensional vector space. In such a space with a metric the vector product Z of vectors X, Y can be constructed by raising the indices of their contraction with a unit 3-form, $Z \equiv X \times Y = g^{-1}(\langle \epsilon, X \otimes Y \rangle)$.

A.3 Scalar product

Definition 5 *The so-called* **covariant metric tensor** $g \in T_2^0$*, which is symmetric, and with components* g_{ij} *in any basis* T^* *which satisfy*

$$\det(g_{ij}) \neq 0 \ ,$$

defines on T *the* **scalar product**

$$(\ \ . \ \ ) : T \times T \to \mathcal{F} \ ,$$

in such a way that $\forall X, Y \in T$

$$(X.Y) = (Y.X) = g(X, Y) = g_{ij} X^i Y^j \tag{A.10}$$

(where X^i *and* Y^j *are components in any dual basis).* g *also defines the* **lowering of indices***, i.e., the mapping*

$$g : T \to T^* \ ,$$

such that

$$\langle g(X) Y \rangle = g(X, Y) \ , \tag{A.11}$$

*and its inverse mapping (***raising of indices***)*

$$g^{-1} : T^* \to T \ ,$$

which defines the scalar product on T^*

$$(\xi.\eta) = (\eta.\xi) = g(g^{-1}(\xi), g^{-1}(\eta)) = g^{ij} \xi_i \eta_j \ , \tag{A.12}$$

where g^{ij} *are components of the* **contravariant metric tensor** $g^{-1} \in T_0^2$*. A basis* $\{e_i\}$ *is* **orthogonal***, if*

$$(e_i.e_j) = 0 \qquad \text{for} \qquad i \neq j \ ,$$

and it is **orthonormal***, if*

$$(e_i.e_j) = \pm \delta_{ij} \ .$$

A.4 Tensor analysis

In the present chapter will be summarised the basic properties of tangent vectors, vector and tensor fields, and their derivatives on curved spaces (manifolds) as is, for example, the case of general-relativistic spacetime.

Definition 6 *A topologic space M is a* **differentiable manifold** *of dimension n, if there is given a so-called complete atlas of coordinate maps, i.e., of the locally defined mappings*

$$\mu, \nu : M \to \mathcal{R}^n \ ,$$

covering the whole M such that (in the region where both are defined) $\mu \circ \nu^{-1}$ is infinitely differentiable with a non-zero jacobian ($\circ$ denotes the composition of the mappings).

Definition 7 *Let M be a differentiable manifold and $\mathcal{F}$ the space of real functions ("scalars") on M. The* **tangent vector** *at the point $A \in M$ is the mapping*

$$X_A : \mathcal{F} \to \mathcal{R} \ ,$$

which is linear, i.e., it satisfies

$$X_A(f + c.g) = X_A(f) + c.X_A(g) \ , \tag{A.13}$$

for any $f, g \in \mathcal{F}$, $c \in \mathcal{R}$, and which satisfies

$$X_A(f.g) = X_A(f).g(A) + f(A).X_A(g) \ . \tag{A.14}$$

The set T_A of all tangent vectors in a fixed point A is called **fibre***, and their unification $B = \cup_{A \in M} T_A$ a* **tangent bundle**.

It can be shown that in a coordinate map μ ($\mu(A) = (x^1_A, ..., x^n_A)$), the tangent vector X_A is given by

$$X_A(f) = X^i_A \frac{\partial}{\partial x^i} (f \circ \mu^{-1})|_{\mu(A)} \ . \tag{A.15}$$

This definition corresponds to the intuitive understanding of a vector X_A as an arrow from the point $A \in M$ to the (infinitesimally distant) point $A + \Delta$ in such a way that $\forall f \in \mathcal{F}$ (and especially for f as one component of the coordinate map) $X_A(f)$ gives the linear term of the Taylor expansion

$$f(A + \Delta) \simeq f(A) + X_A(f) \ .$$

On the fibre T_A there is defined an addition and a multiplication by a number, with respect to which T_A is an n-dimensional vector space and $\{\frac{\partial}{\partial x^i_A}|^n_{i=1}\}$ is its coordinate basis induced by the map μ. The tangent bundle B forms a $2n$-dimensional manifold, the atlas of which is formed by all maps compatible with the map which attributes $\forall X_A \in T_A$ (i.e., to the point A and the vector X_A in it) the coordinates $(x^1_A, ..., x^n_A, X^1_A, ..., X^n_A) \in \mathcal{R}^{2n}$.

Definition 8 *Let M be a differentiable manifold and $\mathcal{F}$ the space of real functions ("scalars") on M. The tangent vector space T on M is the space of* **tangent vectors** *X (or "contravariant vector fields") on M, i.e., the space of mappings*

$$X : \mathcal{F} \to \mathcal{F} \ ,$$

which are linear and differentiable, i.e. satisfying

$$X(f + c.g) = X(f) + c.X(g) \ , \tag{A.16}$$

$$X(f.g) = X(f).g + f.X(g) \ , \tag{A.17}$$

for any $f, g \in \mathcal{F}$, $c \in \mathcal{R}$.

The definition region $\{A \in M\}$ of a vector field can be restricted to some $N \subset M$ ($X : \mathcal{F}(M) \to \mathcal{F}(N)$), especially also to a single point. If N is a submanifold of M, $T(N)$ is then a subspace of $T(M)$ called the space of vectors tangent to N. Obviously, any function on M is simultaneously a function on N, and thus each tangent vector on N is also a vector on M, defined only in points of N and 'insensitive' to a different behaviour of a function apart from N. Each coordinate map $\mu(A) = \{x^i(A)|_{i=1}^n\}$ induces in its definition region a coordinate basis $\{\partial_{x^i}|_{i=1}^n\}$ (here it is understood that $\partial_{x^i} f = \frac{\partial}{\partial x^i}(f \circ \mu^{-1})$). For the change to coordinates $\{y^i\}$ the elements of the coordinate basis are transformed according to

$$\partial_{x^i} = \frac{\partial y^j}{\partial x^i} \partial_{y^j} \ . \tag{A.18}$$

If the tangent vectors are defined as a kind of mappings of the space of scalars into the same space, there is a natural question whether their composition is also a tangent vector. It is obviously linear, however, it does not satisfy the condition (A.17), because

$$X \circ Y(f.g) = X(f.Y(g) + Y(f).g) = f.X \circ Y(g) + X \circ Y(f).g + X(f).Y(g) + Y(f).X(g) \ , \tag{A.19}$$

where the last two terms violate the condition. Subtracting from this equation the same with interchanged X and Y, these additional terms cancel. We have thus proved the following theorem (and definition at the same time):

Theorem 1 *The commutator*

$$[X,Y] = X \circ Y - Y \circ X \tag{A.20}$$

of vectors X, Y is also a vector.

The elements of a coordinate basis commute (they are 'holonomic'). In a general basis $\{e_i\} = \{\theta^i\}^*$, the action of the commutator on vectors expressed by their components can be written as

$$[X,Y] = (X(Y^k) - Y(X^k) + X^i Y^j c^k{}_{ij}) e_k \ , \tag{A.21}$$

where

$$c^k{}_{ij} = -c^k{}_{ji} = \theta^k([e_i, e_j]) \tag{A.22}$$

are the so-called **coefficients of structure** of the given basis.

Definition 9 *A cotangent vector space T^* is the dual vector space of* **1–forms** *or "covariant vector fields", i.e., the space of mappings*

$$\xi : T \to \mathcal{F} \ .$$

By analogy, the space T^p_q of **tensors** *of type p, q is the space of multilinear mappings*

$$A : \underbrace{T^* \times ... \times T^*}_{p} \times \underbrace{T \times ... \times T}_{q} \to \mathcal{F} \ ,$$

*and spaces of p-***vectors** *or p-***forms** *are the antisymmetric subspaces T^p_0 or T^0_p, resp. In the spaces of all tensors or p-vectors and p-forms there are defined the tensor product and the contraction or the external and inner product, resp. in analogy to Definitions 2 and 3.*

A tensor of type T^0_p on M is simultaneously a tensor of the same type on a submanifold $N \subset M$. Especially, a p-form on M is a p-form on N, which can be non-zero only if $p \leq \dim N$.

Definition 10 *An* **exterior derivative** *is the linear mapping*

$$d : \wedge^p T^* \to \wedge^{p+1} T^* \ ,$$

such that $\forall f \in \wedge^0 T^* = \mathcal{F}$

$$df(X) = X(f) \ , \tag{A.23}$$

$\forall \omega \in \wedge^p T^*$

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta) \ , \tag{A.24}$$

and

$$dd\omega = 0 \ . \tag{A.25}$$

For any coordinate map $\{x^i|_{i=1}^n\}$ the exterior derivatives $\{dx^i|_{i=1}^n\}$ form a basis of T^*, which transforms according to

$$dx^i = \frac{\partial x^i}{\partial y^j} dy^j \ , \tag{A.26}$$

and is dual with respect to the coordinate basis $\{\partial_{x^i}|_{i=1}^n\}$ of T.

Theorem 2 *For a 1-form* ω*, its exterior derivative satisfies*

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \ . \tag{A.27}$$

Proof: According to (A.21), in any coordinate basis (where ${c^k}_{ij}$ vanishes and $\omega = \omega_j dx^j$), it is valid that

$$\begin{aligned} X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) &= X(\omega_j Y^j) - Y(\omega_j X^j) - \omega_j (X(Y^j) - Y(X^j)) = \\ &= X(\omega_j) Y^j - Y(\omega_j) X^j = (d\omega_j \wedge dx^j)(X,Y) = d\omega(X,Y) \ . \end{aligned}$$

Q.E.D.

Consequently, alternatively to (A.22) the coefficients of structure can be expressed as

$${c^k}_{ij} = -d\theta^k(e_i, e_j) \ . \tag{A.28}$$

Definition 11 $\omega \in \wedge^p T^*$ *defines the* **measure** *(i.e., the integration) on a p-dimensional submanifold* $N \subset M$

$$\int \omega = \int \omega(\partial_{x^1}, ..., \partial_{x^p}) dx^1 ... dx^p \ , \tag{A.29}$$

where $\{x^i|_{i=1}^p\}$ *is an arbitrary coordinate map on* N*.*

Theorem 3 Stokes theorem *For a closed p-dimensional boundary* ∂V *of a* $(p+1)$*-dimensional region* V*, it is valid that*

$$\int_V d\omega = \int_{\partial V} \omega \ . \tag{A.30}$$

A.5 Affine connection

Definition 12 *An* **affine connection** *is the mapping*

$$\nabla : T \times T \to T \ ,$$

such that $\forall X, Y, Z \in T, f \in \mathcal{F}$

$$\nabla_{f.Y+Z} X = f.\nabla_Y X + \nabla_Z X; , \tag{A.31}$$

$$\nabla_X (f.Y + Z) = X(f).Y + f.\nabla_X Y + \nabla_X Z \ . \tag{A.32}$$

An affine connection $\nabla_X Y$ thus has a tensor character with respect to the index X and the differential behaviour with respect to the argument Y, i.e., it describes the change of the field Y along the field X. In any dual basis $\{e_i\} = \{\theta^i\}^*$,

$$\nabla_X Y = \left(X(Y^k) + Y^i {\omega^k}_i(X) \right) e_k \ , \tag{A.33}$$

where

$${\omega^k}_i = {\omega^k}_{ij} \theta^j = \theta^k (\nabla_{e_j} e_i) \theta^j \tag{A.34}$$

are the **1-forms of connection** in the corresponding basis (their components ${\omega^k}_{ij}$ are the so-called **Ricci rotation coefficients**).

By the requirement

$$X(\langle \xi, Y \rangle) = \langle \nabla_X \xi, Y \rangle + \langle \xi, \nabla_X Y \rangle \ , \tag{A.35}$$

or by the analogous requirement on the derivative of the tensor product and its contraction, there is also defined the affine connection for 1-forms by the relation

$$\nabla_X \xi = \left(X(\xi_k) - \xi_i {\omega^i}_k(X) \right) \theta^k \ , \tag{A.36}$$

or for tensors T^p_q as the mapping

$$\nabla : T \times T^p_q \to T^p_q \ ,$$

such that $\forall A \in T^p_q, X, Y, ... \in T, \xi, ... \in T^*$

$$X(A(\xi, ..., Y, ...)) = \nabla_X A(\xi, ..., Y, ...) + A(\nabla_X \xi, ..., Y, ...) + ... + A(\xi, ..., \nabla_X Y, ...) + ... \ . \tag{A.37}$$

Definition 13 *The* **covariant derivative** *(corresponding to the affine connection ∇) is the mapping*

$$D : T^p_q \to T^p_{q+1} \ ,$$

such that $\forall A \in T^p_q, X \in T$

$$\langle DA, X \rangle = \nabla_X A \ . \tag{A.38}$$

In any dual basis $\{e_i\} = \{\theta^i\}^*$ it is useful to denote

$$DA = \nabla_{e_k} A \otimes \theta^k = A^{i_1,...,i_p}_{j_1,...,j_q;k} e_{i_1} \otimes ... \otimes e_{i_p} \otimes \theta^{j_1} \otimes ... \otimes \theta^{j_q} \otimes \theta^k \ , \tag{A.39}$$

and to write

$$A^{i_1,...,i_p}_{j_1,...,j_q;k} = A^{i_1,...,i_p}_{j_1,...,j_q,k} + A^{i,...,i_p}_{j_1,...,j_q} {\omega^{i_1}}_{ik} + ... - A^{i_1,...,i_p}_{j,...,j_q} {\omega^j}_{j_1 k} - ... \ , \tag{A.40}$$

where

$$A^{i_1,...,i_p}_{j_1,...,j_q,k} = e_k (A^{i_1,...,i_p}_{j_1,...,j_q}) \ .$$

The matrix of 1-forms of connection (see (A.34)) does not behave as a tensor in the indices k and i for a transition to another basis, however, it is possible to produce a tensor from it.

Theorem 4 *If ∇ is an affine connection, then there exists a* **tensor of torsion** *(determined by this connection) $Q \in (T \otimes \wedge^2 T^*) \subset T^1_2$ such, that $\forall X, Y \in T$*

$$\nabla_X Y - \nabla_Y X - [X,Y] = \langle Q, X \otimes Y \rangle \tag{A.41}$$

and a **tensor of curvature** *$R \in (T \otimes T^* \otimes \wedge^2 T^*) \subset T^1_3$, such that $\forall X, Y, Z \in T$*

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \langle R, Z \otimes X \otimes Y \rangle \ . \tag{A.42}$$

To proove this theorem, we must check that there mutually cancel additional terms like $X(f)$, arising from both the covariant derivatives and the commutator if $Y \to fY$ is substituted. The linearity with respect to the addition is obvious.

Both these tensors are antisymmetric in the last two arguments (X and Y). It is thus possible to represent them in a chosen basis by 2-forms of torsion τ^k or curvature $\Omega^k{}_i$ resp., according to the relations

$$Q = e_k \otimes \tau^k \ , \tag{A.43}$$
$$R = e_k \otimes \theta^i \otimes \Omega^k{}_i \ . \tag{A.44}$$

Comparing (A.33) and (A.21), it is obvious that

$$\nabla_X Y - \nabla_Y X - [X,Y] = \left(Y^i \omega^k{}_i(X) - X^i \omega^k{}_i(Y) - X^i Y^j c^k{}_{ij} \right) .e_k \ , \tag{A.45}$$

so that

$$Q^k{}_{ij} = \omega^k{}_{ji} - \omega^k{}_{ij} - c^k{}_{ij} \ . \tag{A.46}$$

Theorem 5 *The 2-forms of torsion are given by the first* **Cartan's equations of structure**, *which read*

$$\tau^k = d\theta^k + \omega^k{}_i \wedge \theta^i \ , \tag{A.47}$$

and the 2-forms of curvature by the second Cartan's eqs. of structure

$$\Omega^k{}_i = d\omega^k{}_i + \omega^k{}_j \wedge \omega^j{}_i \ . \tag{A.48}$$

Specially, for a connection without torsion, the right-hand side of (A.47) is identically equal to zero. The proof of Eq. (A.47) follows from the substitution of (A.28) into (A.46). Similarly (A.48) follows from (A.42) by substituting (A.33) and using (A.27).

Definition 14 *An affine connection ∇ which preserves a scalar product, i.e., which satisfies*

$$Z(X,Y) = (\nabla_Z X, Y) + (X, \nabla_Z Y) \tag{A.49}$$

for all X, Y, Z, is named a **metric connection**.

Inserting here the basis vectors, we arrive at the expression

$$dg_{ij} = \omega_{ij} + \omega_{ji} \ , \tag{A.50}$$

where the 1-forms $\omega_{ij} = g_{ik}\omega^k{}_j$. Rewriting this equation in components

$$e_k(g_{ij}) = \omega_{ijk} + \omega_{jik} \ , \tag{A.51}$$

and using Eq. (A.46) with lowered index k, we can solve for ${\omega^i}_j$, i.e.,

$$\begin{aligned} \omega_{ijk} &= \frac{1}{2}(e_k(g_{ij}) + e_j(g_{ik}) - e_i(g_{jk}) \\ &\quad -c_{ijk} + c_{jik} + c_{kij} \\ &\quad -Q_{ijk} + Q_{jik} + Q_{kij}\ ) \ . \end{aligned} \tag{A.52}$$

An affine connection is thus uniquely determined by requirements of being metric and torsion free (i.e., the third line of this expression being identically equal to zero). The first line of this expression, i.e., the components ${\omega^i}_{jk}$ in a coordinate basis, where also the second line of this expression vanishes, is named Christoffel symbols (usually denoted as ${\Gamma^i}_{jk}$).

Theorem 6 *For a metric connection, the components ${R^i}_{klm} = \langle {\Omega^i}_k, e_l \otimes e_m \rangle$ of the tensor of curvature safisfy the antisymmetry*

$$R_{iklm} = -R_{kilm} \ . \tag{A.53}$$

For a torsion-free connection they satisfy the symmetry

$${R^i}_{(klm)} \equiv {R^i}_{klm} + {R^i}_{lmk} + {R^i}_{mkl} = 0 \ , \tag{A.54}$$

and the **Bianchi identities**

$${R^i}_{k(lm;n)} \equiv {R^i}_{klm;n} + {R^i}_{knl;m} + {R^i}_{kmn;l} = 0 \ . \tag{A.55}$$

In an n-dimensional manifold the **Riemann tensor***, i.e., the tensor of curvature determined by the torsion-free metric connection, thus has $\frac{1}{12}n^2(n^2-1)$ algebraically independent components.*

The **Ricci tensor** *$R_{km} = {R^i}_{kim}$ is then symmetric.*

Theorem 7 *The tangent vector $X = \frac{d}{dt}$ of a line $x = x(t)$ which is an extremal of the length S (i.e., the line is a geodesic),*

$$0 = \delta S = \delta \int \sqrt{g(X,X)} dt \ , \tag{A.56}$$

is parallely transported along the line, i.e.,

$$\nabla_X X = 0 \ . \tag{A.57}$$

Proof: Equations for extremals of an action $S = \int L(x,\dot{x})dt$ are obtained via a variation δx of the path and its derivatives $\delta \dot{x}$. To get the Lagrangian of the geodesics in this form, it is thus advantageous to express its tangent vector X in a coordinate basis of any coordinate system, i.e.,

$$L(x,\dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} \ . \tag{A.58}$$

Inserting this Lagrangian into the standard procedure

$$0 = \delta S = \int \delta L(x,\dot{x}) dt = \int \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \right] \delta x dt \ , \tag{A.59}$$

we arrive at the equations

$$0 = \frac{1}{2\sqrt{g(\dot{x},\dot{x})}} g_{ij,k}(x)\dot{x}^i\dot{x}^j - \frac{d}{dt}\left(\frac{g_{ik}\dot{x}^i + g_{kj}\dot{x}^j}{2\sqrt{g(\dot{x},\dot{x})}} \right) \ , \tag{A.60}$$

i.e.,

$$0=\frac{1}{2}(g_{ij,k}-g_{ik,j}-g_{kj,i})\dot{x}^i\dot{x}^j-g_{ik}\ddot{x}^i-g_{ik}\dot{x}^i\sqrt{g(\dot{x},\dot{x})}\frac{d}{dt}\left(\frac{1}{\sqrt{g(\dot{x},\dot{x})}}\right) . \tag{A.61}$$

This is valid for any parameterization of the path $x(t)$. If we impose the condition of t being the affine parameter, i.e., $t=S$, then $g(\dot{x},\dot{x})=1$ and the last term vanishes. With this condition, the geodesic can thus be calculated equivalently from the variational principle

$$0=\delta\int g(\dot{x},\dot{x})dt . \tag{A.62}$$

Comparing the first term with Eq. (A.52), we find (in view of $c_{ijk}=0$ in coordinate basis), that it corresponds to $-\omega_{kij}\dot{x}^i\dot{x}^j$ of a torsion-free ($Q_{ijk}=0$) metric connection, and the first two terms thus give the right-hand side of Eq. (A.57).

A.6 Lie derivatives and Killing vectors

Definition 15 *Let the* **Lie derivative** $\pounds_X$ *with respect to a vector field* X *be a mapping* $\pounds_X : T^p_q \to T^p_q$ *such that*

$$\pounds_X(A\otimes B)=(\pounds_X A)\otimes B + A\otimes(\pounds_X B) , \tag{A.63}$$

$$\pounds_X\langle A\rangle=\langle\pounds_X A\rangle , \tag{A.64}$$

$$\pounds_X f = X(f) , \qquad \text{for } f\in T^0_0 , \tag{A.65}$$

and

$$\pounds_X Y=[X,Y] , \qquad \text{for } Y\in T^1_0 . \tag{A.66}$$

Obviously, comparing with Eq. (A.31), there is an additional third term in the expression

$$\pounds_{fX+Z}Y=f\pounds_X Y+\pounds_Z Y - Y(f)X . \tag{A.67}$$

Consequently, in a basis $\{e_i\}$

$$\pounds_X Y = X^i\pounds_{e_i}Y - Y(X^i)e_i = X^i e_i(Y^k)e_k - Y^k e_k(X^i)e_i + X^iY^k[e_i,e_k] , \tag{A.68}$$

specially in a coordinate basis

$$(\pounds_X Y)^i = X^kY^i{}_{,k} - Y^kX^i{}_{,k} . \tag{A.69}$$

Because for $\sigma\in T^0_1$

$$\pounds_X\langle\sigma,Y\rangle=\langle\pounds_X\sigma,Y\rangle+\langle\sigma,\pounds_X Y\rangle , \tag{A.70}$$

for the components of a covector Lie-derivative

$$\begin{aligned}(\pounds_X\sigma)_i &= \langle\pounds_X\sigma,e_i\rangle=\pounds_X\langle\sigma,e_i\rangle-\langle\sigma,\pounds_X e_i\rangle = X(\sigma_i)-\langle\sigma,[X,e_i]\rangle\\ &= X^k e_k(\sigma_i)+e_i(X^k)\sigma_k - X^k\langle\sigma,[e_k,e_i]\rangle ,\end{aligned} \tag{A.71}$$

and in a coordinate basis

$$(\pounds_X\sigma)_i = X^k\sigma_{i,k}+\sigma_k X^k{}_{,i} . \tag{A.72}$$

For Lie-derivatives of higher tensors, in addition to the derivative of the component, there appears for each contravariant or covariant index a term with derivatives of X analogous the second term of Eq. (A.69) or Eq. (A.72), resp. If the manifold is Riemannian, the partial derivatives in these equations can also be replaced by the contravariant derivatives, because the additional terms with connections mutually cancel due to their symmetry $\omega_{i(jk)}$.

Definition 16 *In a Riemannian manifold with a metric g, a vector ξ is named a* **Killing vector** *if*

$$\pounds_\xi g = 0\,. \tag{A.73}$$

Expressing this condition in coordinate components

$$0 = (\pounds_\xi g)_{ik} = \xi^l g_{ik;l} + g_{lk}\xi^l{}_{;i} + g_{il}\xi^l{}_{;k}\ , \tag{A.74}$$

we arrive at the condition

$$0 = \xi_{k;i} + \xi_{i;k}\ , \tag{A.75}$$

because the covariant derivatives of the metric components vanish identically for the metric connection.

Theorem 8 *If $X = \frac{d}{dt}$ is a tangent vector to a geodesic, than for any Killing vector ξ the scalar product $g(\xi, X)$ is constant along the geodesic.*

Proof:

$$\frac{d}{dt}g(\xi, X) = (\xi_k X^k)_{;i}X^i = \xi_{k;i}X^i X^k + \xi_k X^k{}_{;i}X^i = 0\ , \tag{A.76}$$

because the first term vanishes due to the antisymmetry $\xi_{[k;i]}$ according to Eq. (A.75), and the second one due to the equation (A.57) of the geodesic.