Chapter 4

Spherically symmetric spacetimes

Solutions of Einstein's field equations with spherical symmetry are one of a few known exact solutions. Being relatively simple, they yield good insight into many features of strong gravitational fields. They include at least a first approximation to two important applications of general relativity — relativistic astrophysics of compact objects and cosmology.

4.1 Geometry in spherical symmetry

The assumption of spherical symmetry of a spacetime means that each event belongs to a two-dimensional submanifold with spherical symmetry, i.e., with the metric

$$ds^2 = R^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \ , \tag{4.1}$$

where ϑ and φ are the standard angular coordinates on the sphere. There must exist (at least locally) two additional coordinates t and r labeling different spheres. They can be chosen to be orthogonal to the spheres and also mutually orthogonal, t to be time-like and r space-like. The metric thus reads

$$g_{\iota\kappa} = \mathrm{diag}\left(-c^2 \mathrm{e}^{2\Phi}, \mathrm{e}^{2\Lambda}, R^2, R^2 \sin^2\vartheta\right) \ , \tag{4.2}$$

where $\Lambda = \Lambda(r,t)$, $\Phi = \Phi(r,t)$ and $R = R(r,t)$. If $\frac{\partial R}{\partial r} \neq 0$, it is possible to rescale r to be identical with R. However, this special choice of the coordinate system need not be useful in particular cases.

The spherically symmetric geometry of spacetime must be produced by a spherically symmetric distribution of mass. Thus, the motion of the mass must also be perpendicular to the spheres of the spacetime symmetry, so that its 4-velocity reads

$$U = \frac{d}{d\tau} = \gamma \mathrm{e}^{-\Phi} \left(\partial_t + u\partial_r\right) \ , \tag{4.3}$$

where τ is the proper time of the medium,

$$u = u(r,t) = \frac{dr}{dt} \tag{4.4}$$

is its coordinate velocity, and the Lorentz factor of the motion of the medium with respect to coordinate tetrads

$$\gamma = \left(1 - \mathrm{e}^{2\Lambda - 2\Phi} \frac{u^2}{c^2}\right)^{-1/2} \ , \tag{4.5}$$

follows from the condition

$$g(U,U) = -c^2 \ . \tag{4.6}$$

To have a simple description of the mass, it is advantageous to use the comoving orthonormal frame

$$\begin{aligned}
e_0 &= \frac{1}{c}U = \frac{\gamma}{c}\mathrm{e}^{-\Phi}\left(\partial_t + u\partial_r\right) \ , &(4.7)\\
e_1 &= \gamma\mathrm{e}^{-\Lambda}\left(\frac{u}{c^2}\mathrm{e}^{2\Lambda-2\Phi}\partial_t + \partial_r\right) \ , &(4.8)\\
e_2 &= \frac{1}{R}\partial_\vartheta \ , &(4.9)\\
e_3 &= \frac{1}{R\sin\vartheta}\partial_\varphi \ , &(4.10)
\end{aligned}$$

where the vectors e_2 and e_3 are tangent to the ϑ- and φ- coordinate lines. The dual basis reads

$$\begin{aligned}
\theta^0 &= c\gamma\mathrm{e}^{\Phi}\left(dt - \frac{u}{c^2}\mathrm{e}^{2\Lambda-2\Phi}dr\right) \ , &(4.11)\\
\theta^1 &= \gamma\mathrm{e}^{\Lambda}\left(-udt + dr\right) \ , &(4.12)\\
\theta^2 &= Rd\vartheta \ , &(4.13)\\
\theta^3 &= R\sin\vartheta d\varphi \ . &(4.14)
\end{aligned}$$

The metric is Minkowskian in these bases, i.e.,

$$g_{\alpha\beta} = g^{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1) \ , \tag{4.15}$$

hence Cartan's first equation of structure (A.47) and the condition (A.50) give the following solution of 1-forms of torsion-free metric connections

$$\begin{aligned}
\omega^0{}_1 = \omega^1{}_0 &= \omega^0{}_{10}\theta^0 + \omega^0{}_{11}\theta^1 \ , &(4.16)\\
\omega^0{}_2 = \omega^2{}_0 &= \frac{1}{R}e_0(R)\theta^2 = \frac{\gamma}{cR}\mathrm{e}^{-\Phi}\left(\partial_t R + u\partial_r R\right)\theta^2 \ , &(4.17)\\
\omega^0{}_3 = \omega^3{}_0 &= \frac{1}{R}e_0(R)\theta^3 = \frac{\gamma}{cR}\mathrm{e}^{-\Phi}\left(\partial_t R + u\partial_r R\right)\theta^3 \ , &(4.18)\\
\omega^1{}_2 = -\omega^2{}_1 &= -\frac{1}{R}e_1(R)\theta^2 = -\frac{\gamma}{R}\mathrm{e}^{-\Lambda}\left(\partial_r R + \frac{u}{c^2}\mathrm{e}^{2\Lambda-2\Phi}\partial_t R\right)\theta^2 \ , &(4.19)\\
\omega^1{}_3 = -\omega^3{}_1 &= -\frac{1}{R}e_1(R)\theta^3 = -\frac{\gamma}{R}\mathrm{e}^{-\Lambda}\left(\partial_r R + \frac{u}{c^2}\mathrm{e}^{2\Lambda-2\Phi}\partial_t R\right)\theta^3 \ , &(4.20)\\
\omega^2{}_3 = -\omega^3{}_2 &= -\frac{\mathrm{cotg}\vartheta}{R}\theta^3 \ , &(4.21)
\end{aligned}$$

where

$$\begin{aligned}
\omega^0{}_{10} &= \mathrm{e}^{-\Lambda-\Phi}\left(\partial_r(\gamma\mathrm{e}^{\Phi}) + \partial_t(\gamma\frac{u}{c^2}\mathrm{e}^{2\Lambda-\Phi})\right) \ , &(4.22)\\
\omega^0{}_{11} &= \frac{1}{c}\mathrm{e}^{-\Lambda-\Phi}\left(\partial_t(\gamma\mathrm{e}^{\Lambda}) + \partial_r(\gamma u\mathrm{e}^{\Lambda})\right) \ . &(4.23)
\end{aligned}$$

The 2-forms of curvature $\Omega^\alpha{}_\beta$ can be obtained from Cartan's second equation of structure (A.48) and Eqs. (4.16) – (4.21) in the form

$$\Omega^1{}_0 = \Omega^0{}_1 = A\theta^0\wedge\theta^1 \ , \tag{4.24}$$

$$
\begin{aligned}
\Omega^2{}_0 = \Omega^0{}_2 &= B\theta^0\wedge\theta^2 + C\theta^1\wedge\theta^2 \quad , && (4.25)\\
\Omega^3{}_0 = \Omega^0{}_3 &= B\theta^0\wedge\theta^3 + C\theta^1\wedge\theta^3 \ , && (4.26)\\
-\Omega^1{}_2 = \Omega^2{}_1 &= C\theta^0\wedge\theta^2 + D\theta^1\wedge\theta^2 \ , && (4.27)\\
-\Omega^1{}_3 = \Omega^3{}_1 &= C\theta^0\wedge\theta^3 + D\theta^1\wedge\theta^3 \ , && (4.28)\\
\Omega^2{}_3 = -\Omega^3{}_2 &= E\theta^2\wedge\theta^3 \ , && (4.29)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= (\omega^0{}_{11})^2 - (\omega^0{}_{10})^2 + e_0(\omega^0{}_{11}) - e_1(\omega^0{}_{10}) \ , && (4.30)\\
B &= \frac{1}{R}\left(e_0(e_0(R)) - \omega^0{}_{10}e_1(R)\right) \ , && (4.31)\\
C &= \frac{1}{R}\left(e_1(e_0(R)) - \omega^0{}_{11}e_1(R)\right) = && (4.32)\\
&= \frac{1}{R}\left(e_0(e_1(R)) - \omega^0{}_{10}e_0(R)\right) \ , \\
D &= \frac{1}{R}\left(e_1(e_1(R)) - \omega^0{}_{11}e_0(R)\right) \ , && (4.33)\\
E &= \frac{1}{R^2}\left(1 + (e_0(R))^2 - (e_1(R))^2\right) \ . && (4.34)
\end{aligned}
$$

Note that both expressions for C are equivalent because of the relation between commutators and connections which follows from Eq. (A.46) for the vanishing torsion. The Ricci tensor

$$
R_{\alpha\beta} = \begin{pmatrix} -A-2B & -2C & 0 & 0 \\ -2C & A-2D & 0 & 0 \\ 0 & 0 & B-D+E & 0 \\ 0 & 0 & 0 & B-D+E \end{pmatrix} \ , \qquad (4.35)
$$

thus gives the scalar curvature

$$
R^\alpha{}_\alpha = 2A + 4B - 4D + 2E \ , \qquad (4.36)
$$

and the Einstein tensor reads

$$
G_{\alpha\beta} = \begin{pmatrix} -2D+E & -2C & 0 & 0 \\ -2C & -2B-E & 0 & 0 \\ 0 & 0 & -A-B+D & 0 \\ 0 & 0 & 0 & -A-B+D \end{pmatrix} \ . \qquad (4.37)
$$

4.2 The Schwarzschild solution

Let us now solve Einstein's field equations (3.29) for the spherically symmetric vacuum case, i.e., the left-hand side given by Eq. (4.37) and the right-hand side being identically equal to zero. The solution will correspond to the outer field of an isolated spherical mass.

In the vacuum there is no medium which we would like to fit by a particular choice of the radial coordinate r or its 4-velocity field U. We can thus fix the solution by the simplifying assumptions $R = r$ and $u = 0$ (despite the fact that we may be interested in a radial motion of a test massive medium, which could justify a non-trivial choice of these variables). In this case Eqs. (4.16), (4.22) and (4.23) can be written simply as

$$
\omega^0{}_1 = \omega^1{}_0 = \mathrm{e}^{-\Lambda}\Phi'\theta^0 + \frac{1}{c}\mathrm{e}^{-\Phi}\dot{\Lambda}\theta^1 \ , \qquad (4.38)
$$

where the dot ($\dot{}$) and the prime ($'$) denote the derivatives with respect to t and r, resp. Also the expressions (4.30) – (4.34) can be simplified as

$$\begin{aligned} A &= \frac{1}{c^2}\mathrm{e}^{-2\Phi}(\ddot{\Lambda}+\dot{\Lambda}^2-\dot{\Lambda}\dot{\Phi})-\mathrm{e}^{-2\Lambda}(\Phi''+\Phi'^2-\Phi'\Lambda')\ , &(4.39)\\ B &= -\frac{1}{r}\mathrm{e}^{-2\Lambda}\Phi'\ , &(4.40)\\ C &= -\frac{1}{cr}\mathrm{e}^{-\Lambda-\Phi}\dot{\Lambda}\ , &(4.41)\\ D &= -\frac{1}{r}\mathrm{e}^{-2\Lambda}\Lambda'\ , &(4.42)\\ E &= \frac{1}{r^2}\left(1-\mathrm{e}^{-2\Lambda}\right)\ . &(4.43)\end{aligned}$$

The component ‘00’ of the Einstein field equation thus reads (cf. Eq. (4.37))

$$E-2D=\frac{1}{r^2}\left(1-\mathrm{e}^{-2\Lambda}\right)+\frac{2}{r}\mathrm{e}^{-2\Lambda}\Lambda'=0\ . \tag{4.44}$$

It can be integrated in the form

$$\ln(1-\mathrm{e}^{-2\Lambda})=\int\frac{2\mathrm{e}^{-2\Lambda}\Lambda' dr}{1-\mathrm{e}^{-2\Lambda}}=-\int\frac{dr}{r}=\ln\frac{2m}{r}\ , \tag{4.45}$$

hence

$$\mathrm{e}^{2\Lambda}=\frac{1}{1-\frac{2m}{r}}\ , \tag{4.46}$$

where the integration constant m could still be a function of t. However, from the component ‘01’ one can see that $C\sim\dot{\Lambda}=0$.

From the sum of components ‘00’ and ‘11’, $-2(B+D)=0$, there follows

$$\Lambda'+\Phi'=0\ , \tag{4.47}$$

hence

$$\mathrm{e}^{2\Phi}=k^2\left(1-\frac{2m}{r}\right)\ , \tag{4.48}$$

where the integration constant k^2 could also be a function of t.[1] For any choice of this function we can rescale the time-like coordinate $t\to t'=t'(t), dt'=kdt$, to fix $k=1$, a choice which corresponds to t being the usual time at the asymptotically (for $r\to\infty$) Minkowskian region of the spacetime.

It can be verified that the identical components ‘22’ and ‘33’, $-A-B+D=0$, are combinations of the previous components and their derivatives. The Schwarzschild metric thus reads

$$ds^2=-c^2\left(1-\frac{2m}{r}\right)dt^2+\frac{dr^2}{1-\frac{2m}{r}}+r^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right)\ . \tag{4.49}$$

It is obvious that for $r\to\infty$ it becomes the Minkowski metric in spherical coordinates.

Comparing the component g_{00} with Eq. (3.19) obtained from the principle of equivalence, we can see that it corresponds to the Newtonian potential of a spherical body with mass $M=\frac{mc^2}{G}$,

[1] One may also arrive at the same result by substituting the solution for Λ into the component ‘11’ and integrating it.

i.e., the integration constant m is the mass of the body in geometric units. We can also compare the spatial components with the weak field approximation (3.50). To get the spatial metric conformally flat[2] (as it is the case of the result (3.50) for the Newtonian approximation), we must first perform the gauge transformation $r \to \tilde{r} = \frac{1}{2}(r - m + \sqrt{r^2 - 2mr})$ to the isotropic coordinates, in which the infinitesimal increase of the circumference of spheres of symmetry is proportional by 2π to their radial distance. In these coordinates the Schwarzschild metric reads

$$ds^2 = -c^2 \left(\frac{1 - m/2\tilde{r}}{1 + m/2\tilde{r}}\right)^2 dt^2 + \left(1 + \frac{m}{2\tilde{r}}\right)^4 \left(d\tilde{r}^2 + \tilde{r}^2 d\vartheta^2 + \tilde{r}^2 \sin^2\vartheta d\varphi^2\right) . \tag{4.50}$$

Obviously, the expansion to terms linear in $\frac{m}{\tilde{r}}$ gives all $\gamma_{\iota\iota} = \frac{2m}{\tilde{r}}$. Consequently, in the linear approximation far away from the centre, the Schwarzschild metric corresponds to the outer gravitational field of a spherical body with mass M.

It can be seen that the components g_{00} and g_{11} of the Schwarzschild metric (4.49) in Schwarzschild coordinates are singular at the Schwarzschild radius $r = 2m$, the first one going to zero and the second one to infinity (and the corresponding contravariant components just the opposite). However, all components of the Riemann tensor (4.24) – (4.29), i.e., the functions (4.39) – (4.43), are non-singular at this radius as it can be verified subsituting here the solution (4.46) and (4.48). It is obvious from the embedding diagram (which is asymptotically planar, bulging in a funnel-like shape closer to the centre and approaching a cylinder $r = 2m$) that the coordinate r fixed by the circumferences of the spheres is here insensitive to the radial distance. On the contrary, all spatial components g_{ij} of the Schwarzschild metric (4.49) expressed in isotropic coordinates are still well-behaved at this point $\tilde{r} = \frac{m}{2}$. However, the time components g_{00} are vanishing in both cases at the Schwarzschild radius. Obviously, the coordinates t and r interchange their meaning of time- and space- like coordinates in the metric (4.49) below this radius. Nonetheless, as in the case of the Rindler spacetime (2.20), which is a part of the Minkowskian spacetime, the vanishing of g_{00} need not mean a real singularity of the spacetime.

It will be shown in Section 4.3 that falling particles can reach the Schwarzschild radius in finite proper time. They can continue the fall below it, and to measure in their neighbourhood non-singular geometry smoothly connected to the outside parts of the spacetime manifold. The apparent singularity is thus an artifact of the choice of reference frames static with respect to the spacetime symmetry and thus radially accelerated (with the proper acceleration infinitely increasing toward the Schwarzschild radius) with respect to the inertial observers freely falling in the gravitational field.

A non-singular coordinate system can be defined also using the radial zero geodesics (i.e., worldlines of photons), for which we can find from (4.49)

$$0 = ds^2 = -c^2 \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} \tag{4.51}$$

and thus

$$c(t - t_0) = \int c dt = \pm \int \frac{r}{r - 2m} dr = \pm r_* \equiv \pm \left[r + 2m \ln(|\frac{r}{2m} - 1|)\right] . \tag{4.52}$$

In the coordinates t, r_* or the corresponding null-coordinates

$$u = ct - r_* , \qquad v = ct + r_* , \tag{4.53}$$

the time-radial part of the metric is conformally flat,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)(c^2 dt^2 - dr_*^2) = -\frac{r - 2m}{r} du dv , \tag{4.54}$$

[2]It means in the form $ds^2 = f(x)\delta_{ij} dx^i dx^j$.

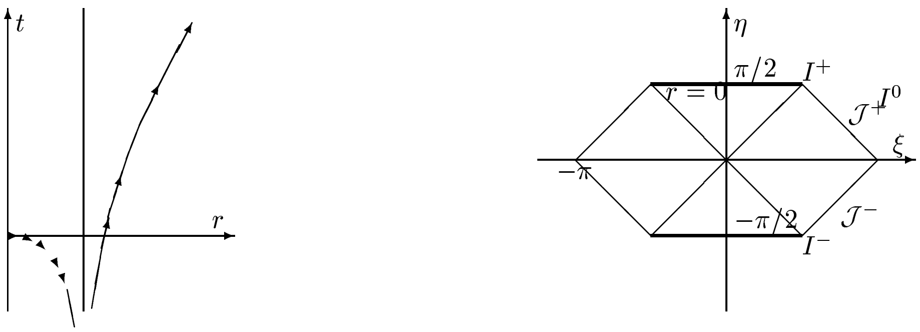

Figure 4.1: Worldline of a radially outgoing photon in Schwarzschild coordinates (left) and Schwarzschild geometry in compactified Kruskal coordinates (right).

but the events of crossing the Schwarzschild radius by the photons are now at infinity in both coordinates ($u \to \infty, v \to -\infty$). It is thus necessary to rescale both these coordinates to bring these events into the finite values of coordinates in such a rate, to cancel the singularity of the conformal factor at $r = 2m$. This singularity should be atributed to the logarithmic term in Eq. (4.52), hence it can be expressed from there as $r - 2m = 2m\exp(r_*/2m - r/2m)$. The metric can thus be written in the form

$$ds^2 = -\frac{2m}{r}\mathrm{e}^{-r/2m}\mathrm{e}^{(v-u)/4m}dudv = -\frac{32m^3}{r}\mathrm{e}^{-r/2m}dUdV\ , \tag{4.55}$$

where

$$U = -\mathrm{e}^{-u/4m} = T - X\ , \qquad V = \mathrm{e}^{v/4m} = T + X\ , \tag{4.56}$$

i.e.,

$$\frac{r-2m}{2m}\mathrm{e}^{\frac{r}{2m}} = -UV = -T^2 + X^2\ , \qquad ct = \frac{u+v}{2} = 2m\ln(-\frac{V}{U}) = 2m\ln\frac{T+X}{X-T}\ . \tag{4.57}$$

These Kruskal coordinates can next be compactified by the conformal transformation

$$T - X = \mathrm{tg}\frac{\eta - \xi}{2}\ , \qquad T + X = \mathrm{tg}\frac{\eta + \xi}{2}\ . \tag{4.58}$$

In the Kruskal coordinates, the Schwarzschild metric is singular only at $r = 0$, i.e., $T^2 - X^2 = 1$. The Schwarzschild radius is splitted into the lines $U = 0$ and $V = 0$. The spacetime region above the Schwarzschild radius is confined in the quadrant $X > |T|$. Time-like worldlines from this region can cross the Schwarzschild radius and continue to the future singularity, but no signal can be sent from this region back. This is why the Schwarzschild radius is also called the Schwarzschild horizon of a black hole.

Exercise 16 *Calculate the proper shape of the embedding diagram of the surface $t =$ constant, $\vartheta = \pi/2$, for the Schwarzschild metric (4.49).*

4.3 Geodesics in spherically symmetric spacetimes

To calculate the geodesic motion or a spacelike geodesic we should solve the second-order differential equations (3.20) with the corresponding connections – in the present case with those given by Eqs. (4.16) – (4.21). However, in agreement with Noether's theorem, the symmetry of spacetime results in a conservation law, i.e., an integral of motion, which will simplify the solution. It can really be seen from the variational principle (A.62), which is equivalent to Eq. (3.20), that if the metric does not depend on a coordinate x^ι (which is thus 'cyclic') the conjugated momentum, which for the geodesic motion is the corresponding covariant component of 4-velocity $g_{\iota\kappa}\dot{x}^\kappa$, is constant (unlike the previous chapter, $\dot{} \equiv \frac{d}{d\tau}$ here). In the spherically symmetric metric (4.2) there is the longitude φ a cyclic coordinate, consequently

$$R^2 \sin^2\vartheta\dot{\varphi} = K \ , \tag{4.59}$$

where K is a constant of motion. However, because the spherical symmetry is two-parametric, there must also be another constant of motion corresponding to the arbitrariness of the orientation of the angular coordinates. Really, the ϑ-component of the variational equation reads

$$0 = \frac{d}{d\tau}(2R^2\dot{\vartheta}) - \frac{\partial}{\partial\vartheta}(R^2\sin^2\vartheta\dot{\varphi}^2) \ , \tag{4.60}$$

which after the substitution for $\dot{\varphi}$ from (4.59) is

$$0 = \frac{d}{d\tau}(2R^2\dot{\vartheta}) + K^2R^{-2}\frac{d}{d\vartheta}(\sin^{-2}\vartheta) \ , \tag{4.61}$$

and can be integrated in the form

$$R^4\dot{\vartheta}^2 = L^2 - K^2\sin^{-2}\vartheta \ , \tag{4.62}$$

where L is the second constant of motion. This constant can be interpreted as the angular momentum

$$L = R^2\dot{\psi} = R^2\sqrt{\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2} \tag{4.63}$$

in the orbital plane, in which the orbit can be parametrized by the angle ψ. The constant K is the projection of the angular momentum to the polar axis of coordinates $\{\vartheta, \varphi\}$. Obviously, ψ can also be taken as a longitude in some spherical coordinates, so that both ∂_φ and ∂_ψ are Killing vectors of the spacetime and Eqs. (4.59) and (4.63) are the corresponding conservation laws of the type (A.76).

If the spacetime (4.2) is also static (i.e., $\partial_t\Phi = \partial_t\Lambda = \partial_t R$) as is the case e.g. of the Schwarzschild metric (4.49), there exists a third Killing vector ∂_t, and the corresponding conservation law reads

$$c\mathrm{e}^{2\Phi}\dot{t} = E \ . \tag{4.64}$$

Finally, because of the fixed parametrization of the geodesic by the affine parameter (or due to the explicit independence of the Lagrangian on the parameter), we have always the integral of motion

$$-\mu = -c^2\mathrm{e}^{2\Phi}\dot{t}^2 + \mathrm{e}^{2\Lambda}\dot{r}^2 + R^2\dot{\psi}^2 \ , \tag{4.65}$$

where $\mu = +1, 0, -1$ for the timelike, null or spacelike geodesics, resp. If we substitute here from Eqs. (4.63) and (4.64), the derivative of the radial coordinate can be expressed in the form

$$\dot{r} = \pm\mathrm{e}^{-\Lambda}\left(\mathrm{e}^{-2\Phi}E^2 - \mu - R^{-2}L^2\right)^{1/2} \ . \tag{4.66}$$

The solution of the geodesic motion in the static spherically symmetric spacetime thus reduces to the solution of a single differential equation of the first order for the radial coordinate (and the consecutive integration of the time and angular coordinates). This can be understood as one-dimensional motion in an effective potential. The first term on the right-hand side of this equation gives the total energy of the particle, the last one the effective centrifugal potential. Comparing the value of E with the shape of the potential, we can study the limits of the radial motion.

Let us investigate this motion in the special case of the Schwarzschild metric. After substitution for Φ and Λ from the solutions (4.48) and (4.46), and fixing $R = r$, the equation reads

$$\dot{r}^2 = E^2 - \left(1 - \frac{2m}{r}\right)\left(\mu + \frac{L^2}{r^2}\right) . \tag{4.67}$$

Exercise 17 *Using the equation for the radial component of geodesic motion in the Schwarzschild spacetime (4.67) calculate the fall (i.e., find the dependence of r on τ) of a particle from the marginally stable circular orbit to the centre. What is the proper time (for $\mu = 1$) needed to cross the Schwarzschild radius and to reach the singularity? Hint: Solve the resulting integral using integration by parts; the substitution $r = x^2$ may be also useful.*

4.4 Spherical solutions with perfect fluid

In this section we will deal with the physically most important solutions of Einstein's field equations, corresponding to the gravitational fields in spacetime filled by matter with the stress-energy tensor given by the perfect-fluid approximation

$$T = \rho U \otimes U + P(g + c^{-2} U \otimes U) , \tag{4.68}$$

where U is the 4-velocity of the fluid, ρ its proper density and P the pressure. Obviously, the first term is the component in the U-direction and the second one is perpendicular to this direction (which is in agreement with the advertised perfectness and isotropy of the fluid), so that in any comoving frame ($e_0 = U/c$), $T^{\iota\kappa} = \mathrm{diag}(\rho c^2, P, P, P)$.

Despite there may be in particular cases reasons for another choice also, let us suppose that we use a comoving radial coordinate r, i.e., in Eqs. (4.7) – (4.10) $u = 0$ and $\gamma = 1$. The expressions (4.30) – (4.34) for the components of 2-forms of curvature (4.24) – (4.29) can then be written in the form

$$A = \frac{1}{c^2}\mathrm{e}^{-2\Phi}(\ddot{\Lambda} + \dot{\Lambda}^2 - \dot{\Lambda}\dot{\Phi}) - \mathrm{e}^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda') , \tag{4.69}$$

$$B = \frac{1}{c^2 R}\mathrm{e}^{-2\Phi}(\ddot{R} - \dot{R}\dot{\Phi}) - \frac{1}{R}\mathrm{e}^{-2\Lambda}\Phi' R' , \tag{4.70}$$

$$C = \frac{1}{cR}\mathrm{e}^{-\Lambda-\Phi}(\dot{R}' - \dot{R}\Phi' - \dot{\Lambda}R') , \tag{4.71}$$

$$D = \frac{1}{R}\mathrm{e}^{-2\Lambda}(R'' - R'\Lambda') - \frac{1}{c^2 R}\mathrm{e}^{-2\Phi}\dot{R}\dot{\Lambda} , \tag{4.72}$$

$$E = \frac{1}{R^2}\left(1 + \frac{1}{c^2}\mathrm{e}^{-2\Phi}\dot{R}^2 - \mathrm{e}^{-2\Lambda}R'^2\right) , \tag{4.73}$$

the first one being identical and the following more general than (4.39) – (4.43), because of the general form of $R = R(t, r)$ expressing the independence of the comoving radial coordinate r and the local radius R of the spheres of space symmetry.

Like in the vacuum case we have again no radial mass flow T^{01}, hence, following Eq. (4.37),

$$C \sim \dot{R}' - \dot{R}\Phi' - \dot{\Lambda}R' = 0 . \tag{4.74}$$

From the isotropy of the stress, $T^{11} = T^{22} = T^{33}$, there follows the relation

$$A = B + D + E . \tag{4.75}$$

In addition to these conditions on the spacetime geometry, we have two non-homogeneous components of the Einstein field equations – '00' for the influence of mass distribution on gravity,

$$E - 2D = 8\pi G c^{-2} \rho \ , \tag{4.76}$$

and '11' for the influence of the pressure,

$$-E - 2B = 8\pi G c^{-4} P \ . \tag{4.77}$$

From these equations there follow the conservation laws (3.28)

$$0 = T^{\iota\kappa}{}_{;\kappa} = e_\kappa(T^{\iota\kappa}) + T^{\lambda\kappa}\omega^\iota{}_{\lambda\kappa} + T^{\iota\lambda}\omega^\kappa{}_{\lambda\kappa} \ , \tag{4.78}$$

i.e., the continuity of mass density

$$0 = e_0(T^{00}) + T^{\kappa\kappa}\omega^0{}_{\kappa\kappa} + T^{00}\omega^\kappa{}_{0\kappa} = \mathrm{e}^{-\Phi}\frac{\partial}{c\partial t}(\rho c^2) + (\rho c^2 + P)\frac{1}{c}\mathrm{e}^{-\Phi}(\dot{\Lambda} + 2\frac{\dot{R}}{R}) \ , \tag{4.79}$$

and the conservation of the radial momentum

$$0 = e_1(T^{11}) + T^{\kappa\kappa}\omega^1{}_{\kappa\kappa} + T^{11}\omega^\kappa{}_{1\kappa} = \mathrm{e}^{-\Lambda}\frac{\partial}{\partial r}(P) + (\rho c^2 + P)\mathrm{e}^{-\Lambda}\Phi' \ . \tag{4.80}$$

The tangential components '22' and '33' express only the independence of P on ϑ and φ. Both the above given equations can be understood in their classical meaning: the temporal change of density is proportional to the divergence of streamlines, i.e., the temporal change of the proper volume of the element of mass, and the gradient of pressure is proportional to the gradient of the gravitational potential, i.e., to the gravitational acceleration (which may contain also the apparent acceleration due to the motion of mass). However, it is interesting to note that, unlike in the Newtonian hydrodynamics, the coefficients of proportionality in these equations contain P in addition to the mass-energy ρc^2. This corresponds to the work performed by the pressure in the course of the element expansion and to the weight of pressure in energy and momentum conservation, resp.

Let us now treat in details two particular applications of these equations for the inner structure of stars and to the dynamics of the universe.

4.4.1 Stellar interiors

Stars are isolated objects which are mostly static or quasistatic if we neglect for now effects like supernova outbursts, stellar pulsations or the secular change of their structure in the course of stellar evolution. The stars can be treated as spherically symmetric despite in individual cases there are observable departures from this assumption (rotational oblateness, tidal distortions in binaries, non-radial pulsations, etc.), which are also worth investigation. Stars are surrounded by almost vacuum external gravitational field, which can be taken as asymptotically Minkowskian at mean interstellar distances and with the above assumption of spherical symmetry, the external field can be taken as the Schwarzschild one from the surface of a single star. Stars are characterized by the outflow of radiative energy and mostly also corpuscular radiation and stellar wind. With these effects there are connected non-zero components T^{01} and also an anisotropy of pressure, $T^{11} \neq T^{22} \simeq T^{33}$. However, this violation of perfectness of the fluid in stellar interiors is mostly negligible, at least as well as the above mentioned effects or others like the magnetic fields.

Let us thus impose on the Einstein's equations (4.76) and (4.77) again (like in the Schwarzschild case, i.e., in the vacuum solution) the assumption of vanishing of all time derivatives. Equations

(4.74) and (4.79) are then identically satisfied. We can choose again the radial coordinate $r \equiv R$. Einstein's equations than read

$$\frac{1}{r^2}\frac{d}{dr}\left[r(1-\mathrm{e}^{-2\Lambda})\right] = 8\pi G c^{-2}\rho \ , \tag{4.81}$$

and

$$\frac{2}{r}\mathrm{e}^{-2\Lambda}\Phi' - \frac{1}{r^2}(1-\mathrm{e}^{-2\Lambda}) = 8\pi G c^{-4} P \ . \tag{4.82}$$

To integrate these equations, it is useful to introduce in analogy with Eq. (4.46) a new radial variable

$$M = M(r) \equiv \frac{c^2}{2G} r(1-\mathrm{e}^{-2\Lambda}) \tag{4.83}$$

(or in geometric units $m(r) = Gc^{-2}M(r)$), which has obviously the meaning of the gravitational mass inside the sphere $r = \mathit{constant}$, as it could be measured by the curvature of the space-like section of the spacetime. The time- component of Einstein's equations gives for this mass the standard (non-relativistic) formula

$$M(r) = \int_0^r 4\pi r^2 \rho dr \ , \tag{4.84}$$

where the integration constant has been determined by the condition of regularity at $r = 0$. If Λ is expressed in terms of $M(r)$ also in Eq. (4.82), it can be treated as an equation for the gravitational potential (time- component of the metric) Φ

$$\Phi' = r\mathrm{e}^{2\Lambda}\left[4\pi G c^{-4} P + \frac{1}{2r^2}(1-\mathrm{e}^{-2\Lambda})\right] = G r \frac{\frac{M(r)}{c^2 r^3} + 4\pi c^{-4} P}{1 - \frac{2GM(r)}{c^2 r}} \ . \tag{4.85}$$

This expression can also be substituted into the equation for the radial momentum (4.80), i.e.,

$$\frac{d}{dr}P = -(\rho c^2 + P)\Phi' = -G\frac{(\rho + c^{-2}P)(M(r) + 4\pi c^{-2} r^3 P)}{r^2(1 - \frac{2GM(r)}{c^2 r})} \ . \tag{4.86}$$

This is the Tolman – Oppenheimer – Volkoff (TOV) equation, i.e., the relativistic equation of hydrostatic equilibrium. The second terms in all three parentheses give the relativistic corrections to the Newtonian formula. Taking into account the proper equation of state $P = P(\rho, T)$ (and a corresponding calculation of the temperature T), one can integrate Eqs. (4.83) and (4.86) from the centre of star to get its structure and overall radius $r_* = r|_{P=0}$ and mass $M(r_*)$. Note, that in Newtonian physics any weight of upper stellar layers can be hold in hydrostatic equilibrium by a sufficiently steep gradient of the pressure, i.e., by the inward increase of P. However, in general relativity the pressure also produces gravity and has its weight. The denominator of this equation can diverge if for any shell inside the star its radius r becomes smaller than the Schwarzschild radius corresponding to its mass $M(r)$.

4.4.2 Homogeneous cosmological models

The physical relation between the distribution of mass and the spacetime geometry in general relativity gives the possibility to investigate the global behaviour of the universe on the basis of some assumptions about its mass contents. The simplest assumption about the mass distribution is the **cosmological principle**, according to which the universe is homogeneous and isotropic after smoothing on some sufficiently large scale.[3] Because this assumption is even more restrictive than the spherical symmetry, we can investigate corresponding models as a special case of spherical solutions with $\rho' = 0 = P'$.

Inserting the assumption $P' = 0$ into Eq. (4.80), one can see that also $\Phi' = 0$ (provided $\rho c^2 + P \neq 0$). It is obvious from the form (4.2) of the metric that by rescaling the coordinate time t to the proper[4] time of the observers comoving with the cosmological fluid, we can set $\Phi = 0$. The expressions (4.69) – (4.73) for the components of 2-forms of curvature can then be simplified to

$$\begin{aligned} A &= \frac{1}{c^2}(\ddot{\Lambda} + \dot{\Lambda}^2) , && (4.87) \\ B &= \frac{1}{c^2 R}\ddot{R} , && (4.88) \\ C &= \frac{1}{cR}\mathrm{e}^{-\Lambda}(\dot{R}' - \dot{\Lambda}R') = \frac{1}{cR}\frac{\partial}{\partial t}(R'\mathrm{e}^{-\Lambda}) , && (4.89) \\ D &= \frac{1}{R}\mathrm{e}^{-2\Lambda}(R'' - R'\Lambda') - \frac{1}{c^2 R}\dot{R}\dot{\Lambda} , && (4.90) \\ E &= \frac{1}{R^2}\left(1 + \frac{1}{c^2}\dot{R}^2 - \mathrm{e}^{-2\Lambda}R'^2\right) . && (4.91) \end{aligned}$$

From the component '01' of Einstein's equation ($C = 0$), one can see that there must exist a function

$$f = f(r) = R'\mathrm{e}^{-\Lambda} . \qquad (4.92)$$

The unknown function Λ and its derivatives can thus be expressed from here as combinations of R, f and their derivatives

$$\mathrm{e}^{\Lambda} = \frac{R'}{f} , \qquad \dot{\Lambda} = \frac{\dot{R}'}{R'} , \qquad \ddot{\Lambda} = \frac{\ddot{R}'}{R'} - \left(\frac{\dot{R}'}{R'}\right) , \qquad \Lambda' = \frac{R''}{R'} - \frac{f'}{f} . \qquad (4.93)$$

The equation of continuity (4.79) can then be written as

$$0 = \dot{\rho}c^2 + (\rho c^2 + P)(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}) , \qquad (4.94)$$

i.e.,

$$-\frac{\dot{\rho}c^2}{\dot{\rho}c^2 + P} = \frac{\partial}{\partial t}\ln(R^2 R') = \frac{\partial}{\partial t}\ln(\frac{1}{3}\frac{\partial}{\partial r}R^3) . \qquad (4.95)$$

Because due to the assumption of homogeneity of P and ρ, the left-hand side is a function of t only, the logarithm on the right-hand side, which is a function of both t and r, must be a sum of two functions of each variable and its argument must be a product of such functions. It means that also R^3 must be such product (plus an integration constant which could depend on t, which, however, must be zero to keep $R(r = 0) = 0$). Consequently, R must be separable in its variables, i.e.,

$$R(r, t) = a(t)x(r) , \qquad (4.96)$$

[3]Let us note, however, that this minimal scale of homogeneity is always observationally shifted behind the limit of the well mapped part of the Universe.

[4]This time is 'Universal', i.e., the same for any of the privileged comoving observers, for which the Universe is homogeneous, but the proper time of other inertial observers moving with respect to the cosmological fluid is related to it by the usual Lorentz transformation.

and the Einstein's field equations (4.76) and (4.77) read

$$\frac{(1-f^2)x' - 2xff'}{a^2x^2x'} + \frac{3\dot{a}^2}{c^2a^2} = \frac{8\pi G}{c^2}\rho \ , \tag{4.97}$$

and

$$-\frac{1-f^2}{a^2x^2} - \frac{2\ddot{a}}{c^2a} - \frac{\dot{a}^2}{c^2a^2} = \frac{8\pi G}{c^4}P \ . \tag{4.98}$$

Because the term $(1-f^2)/x^2$ is the only function of r in the latter equation, it must be a constant

$$\frac{1-f^2}{x^2} = k \ . \tag{4.99}$$

This is consistent also with Eq. (4.97), where

$$\frac{(1-f^2)x' - 2xff'}{x^2x'} = \frac{\frac{d}{dr}(1-f^2)x}{x^2x'} = \frac{\frac{d}{dr}kx^3}{x^2x'} = 3k \ . \tag{4.100}$$

Einstein's equations can thus be separated into Eq. (4.99) for the geometry of the space slices of the homogeneous universe, and into equations

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \ , \tag{4.101}$$

and

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}P \ , \tag{4.102}$$

for the dynamics of the universe.

In view of Eqs. (4.92), (4.96) and (4.99), the spacetime interval (i.e., the metric (4.2)) now reads

$$\begin{aligned} ds^2 &= -c^2dt^2 + a^2(t)\left[\left(\frac{x'(r)}{f(r)}\right)^2 dr^2 + x^2(r)\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right] = \\ &= -c^2dt^2 + a^2(t)\left[d\chi^2 + \Sigma^2(\chi)\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right] \ , \end{aligned} \tag{4.103}$$

where

$$\chi = \int \frac{x'dr}{f} = \int \frac{dx}{\sqrt{1-kx^2}} \tag{4.104}$$

is a new radial coordinate with the meaning of a 'polar distance' from the chosen origin of comoving spherical coordinates, and

$$\Sigma(\chi) \equiv x = \begin{cases} \sin\chi & \text{for} \quad k = 1 \ , \\ \chi & \text{for} \quad k = 0 \ , \\ \mathrm{sinh}\chi & \text{for} \quad k = -1 \ , \end{cases} \tag{4.105}$$

is the inverse function with the meaning of a 'radius of parallel' in these coordinates. Note that originally in Eq. (4.99) k was any real number, but because in Eq. (4.96) the functions a and x can be rescaled by any inversely proportional multiplicative factor, we can ever rescale $x \to \tilde{x} = \sqrt{|k|}x$, so that only the sign of k is important. The values $+1, 0, -1$ of k are the scalar curvature of the space $ds^2 = d\chi^2 + \Sigma^2(\chi)\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$, and they correspond to the closed, marginally opened and opened universe, resp. This geometry of the space in a homogeneous universe is usually postulated as the only choice of a homogeneous and isotropic geometry, and the Einstein's field equations then give directly Eqs. (4.101) and (4.102) for the dynamics of the only free function $a(t)$. However, in agreement with Mach's principle, also the geometry of space is determined in the above given way by the homogeneous distribution of sources.

The dynamics can again be treated as one-dimensional motion, for which Eq. (4.101) is the energy conservation; the first term being the kinetic energy, the second one the potential energy of space curvature, and on the right-hand side the gravitational bounding energy. The density obeys the equation of conservation (4.94) which now reads

$$0 = \dot{\rho}c^2 + 3\frac{\dot{a}}{a}(\rho c^2 + P) . \tag{4.106}$$

In the case of a matter-dominated universe where $P = 0$, these equations are identical to those of Newtonian cosmology, i.e.,

$$\frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3}\frac{\rho a^3}{a} = -\frac{kc^2}{2} , \tag{4.107}$$

where the first term on the left-hand side is the kinetic energy of a particle moving with the surface of an expanding sphere of radius $a(t)$, the second one is its potential energy in the gravitational field of this sphere with mass $M = \frac{4\pi}{3}\rho a^3$, which is constant during the motion, and finally on the right-hand side is the total energy, which is positive for $k = -1$, and thus corresponds to the ever-lasting expansion, and it is negative for $k = +1$, when the expansion stops at a maximum radius and turns into a collapse. For a sufficiently small value of a the right-hand side is negligible and the dynamics is thus similar to the marginal case of $k = 0$ when $a \sim t^{2/3}$, i.e., the expansion reaches asymptotically rest at infinity.

Another important case is the radiation-dominated universe with pressure $P = \frac{1}{3}\rho c^2$, corresponding e.g., to photon gass or an ultrarelativistic gass. Following Eq. (4.106) then $\rho \sim a^{-4}$ (i.e., the temperature T of black-body radiation satisfies $\sigma T^4 \sim \rho \sim a^{-4}$), and for $k = 0$ or at small a the dynamics is $a \sim t^{1/2}$. In a mixture of non-interacting radiation and non-relativistic gass the radiation dominates at smaller radii a.

Finally, at a very early stages of the universe there are supposed to be important equations of state with $P = -\rho c^2$, for which, following Eq. (4.106), $\rho = constant$, i.e., its 'vacuum' density is held constant by the work of expansion against the negative pressure. In this case $a \sim \exp(Ht)$, where $H = \dot{a}/a = constant$. Such a component of matter with a stress-energy tensor $T \sim g$ is equivalent to a cosmological constant, which was originally introduced by Einstein to balance the universe (which was supposed to be static) against the gravitational attraction of usual matter.

For the theory of observations in an expanding universe, the solution of null geodesics $cdt = a(t)d\chi$ is important. If there exists finite $\int_0^t a^{-1}(t)dt$, no signal from behind the corresponding particle horizon could reach the event at time t. Its limit for $t \to t_{max}$ is the so-called event horizon.