Chapter 3

Basic concepts of general relativity

In this chapter will be summarized the basic concepts of the general relativity. First, in Section 3.1, some reasons for the use, and illustrations of the consequences of differential geometry will be presented (differential geometry is reviewed in Appendix A). Together, this section and Appendix A thus correspond to chapters 2 and 3 (and appendix B) of [6]. In the following Sections (corresponding mainly to chapter 4 of [6]) will be explained the physical motivation for identifying the gravitational field with the geometry of spacetime, the formulation of the interaction of gravity with other physical fields and particles, and finally its correspondence with the Newton theory will be proven.

3.1 Geometry of curved spacetime

The concept of manifold (Definition 6) is used to handle effectively the natural topology of spacetime (or other physical spaces). General relativity forces us to limit the use of coordinate systems only as names (and arrangement) of events, and to treat the distances as an additional structure on the spacetime. However, even in flat spacetime or in nonrelativistic physics it is often advantageous to use some curvilinear coordinates to fit the symmetry of the problem in question, despite that the expression for distance will then be more complicated (cf. Cartesian vs. geographic spherical coordinates on the Earth).

Because of this limited significance of coordinates, it is advantageous to define all geometrical objects independently of the existence and choice of a particular coordinate system. This is the case of Definition 7 of vectors as differential operators. Having this definition of vectors, we arrive at their natural expression in any coordinate system

$$X = X^i \frac{\partial}{\partial x^i} = X'^j \frac{\partial}{\partial x'^j} \tag{3.1}$$

and the transformation properties of their components (cf. Eq. (A.18) for the relation of bases)

$$X^i = X'^j \frac{\partial x^i}{\partial x'^j} \; . \tag{3.2}$$

In the classical approach, quantities with components obeying just these transformation rules are defined as the 'contravariant vectors' unlike the so-called 'covariant vectors' obeying the rule (cf. Eq. (A.26))

$$\omega_i = \omega'_j \frac{\partial x'^j}{\partial x^i} \; . \tag{3.3}$$

However, to avoid a confusion, it is better to keep in mind that these are the co-vectors or 1-forms, i.e., elements of the dual vector space (see Definition 1). They can be spanned by exterior derivatives (differentials) of scalars – see Eq. (A.23) in Definition 10. Differentials dx^i of coordinates form a natural coordinate-induced basis, which is dual to the partial derivatives $\frac{\partial}{\partial x^i}$,

$$\omega = \omega_i dx^i \tag{3.4}$$

(from which the above transformation rule of covariant components easily follows).

Following Definition 5, there is a one-to-one correspondence between vectors and covectors defined by the scalar product (i.e., by the metric). In the physics (and geometry) of spacetime and other spaces of interest, we usually have a fundamental metric with respect to which we can identify these spaces and to take contravariant and covariant components as different expressions of the same physical quantity. It is important to remember that the values of these components can differ (in pseudo-Riemannian spaces at least by sign even in orthonormal bases). This is why upper and lower indices are distinguished in general relativity. As an example we can take the 4-velocity $U \equiv \frac{d}{d\tau}$ which has contravariant components $U^\iota = (\gamma, \gamma v^i)$ in the coordinate basis $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\}$ and covariant $U_\iota = (-\gamma c^2, \gamma v^i)$ in the dual basis $\{dt, dx^i\}$ in which $ds^2 = -c^2dt^2 + (dx^i)^2$. There are physical problems for which it is reasonable to introduce some additional metrics. The physical meaning of vectors and covectors and their correspondence must be then investigated carefully.

In practice, it is often advantageous to use a basis other than the coordinate bases both for vectors and covectors. For instance, in the geographic coordinates φ, λ, in which the distance element ds reads (on a spherical Earth)

$$d^2s = r^2(d^2\varphi + \cos^2\varphi d^2\lambda) \ , \tag{3.5}$$

we use to express the vectors of ship velocities (i.e., derivatives with respect to their time) in east-west and north-south components V^1, V^2, resp., in the same units, i.e., with respect to an orthonormal basis

$$e_1 = \frac{1}{r}\frac{\partial}{\partial\varphi} \ , \qquad e_2 = \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} \ , \tag{3.6}$$

or even in polar coordinates in the tangent space, i.e., by the azimuth and the absolute value $|V|$ (e.g., in knots), which can be calculated from the components using the simple metric

$$g_{ij} = \mathrm{diag}(1,1) \ . \tag{3.7}$$

The basis vectors need not commute (in the sence of Eq. (A.20), e.g., in our case

$$[e_1, e_2] = [\frac{\partial}{r\partial\varphi}, \frac{\partial}{r\cos\varphi\partial\lambda}] = \frac{\sin\varphi}{r^2\cos^2\varphi}\frac{\partial}{\partial\lambda} = \frac{\tan\varphi}{r}e_2 \ , \tag{3.8}$$

but they are more appropriate to the local physics. The dual basis to (3.6) reads

$$\theta^1 = r.d\varphi \ , \qquad \theta^2 = r\cos\varphi.d\lambda \ , \tag{3.9}$$

and its exterior derivatives

$$d\theta^1 = 0 \ , \qquad d\theta^2 = -r\sin\varphi d\varphi \wedge d\lambda = -\frac{\tan\varphi}{r}\theta^1 \wedge \theta^2 \ . \tag{3.10}$$

Because the components of the metric tensor in the orthonormal basis are constant, it follows from Eq. (A.50) that

$$\omega_{11} = \omega_{22} = 0 \ , \quad \text{and} \quad \omega_{12} = -\omega_{21} \ . \tag{3.11}$$

Following the first equation of structure (A.47), the 1-forms of the torsion-free metric connection thus satisfy

$$0 = \omega_{12} \wedge \theta^2 \ , \qquad 0 = -\frac{\tan\varphi}{r}\theta^1 \wedge \theta^2 + \omega_{21} \wedge \theta^1 \ . \tag{3.12}$$

It is obvious from the first of these equations that ω_{12} must be proportional to θ^2 only, and from the second one we thus find

$$\omega_{12} = -\omega_{21} = -\frac{\tan\varphi}{r}\theta^2 \ . \tag{3.13}$$

The equations of the geodesics thus read

$$\frac{dV^1}{dt} = \frac{\tan\varphi}{r}V^2V^2 \ , \qquad \frac{dV^2}{dt} = -\frac{\tan\varphi}{r}V^1V^2 \ . \tag{3.14}$$

It can be verified from here that the value of the velocity is constant. Following the second equation of structure (A.48), the 2-forms of curvature on the globe are

$$\Omega_{ij} = \epsilon_{ij} d\omega_{12} = \epsilon_{ij} d(\sin\varphi d\lambda) = \epsilon_{ij}\cos\varphi d\varphi \wedge d\lambda = \epsilon_{ij}\frac{1}{r^2}\theta^1 \wedge \theta^2 \ , \tag{3.15}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$ are components of a unit antisymmetric matrix. This result is consistent with the fact that the curvature is constant on a sphere.

In general relativity, similar orthonormal frames correspond to local inertial observers – they lay out in the spacetime the directions of the observer's proper time and space. It is thus advantageous to describe the physical quantities which are of vector or tensor character with respect to just these frames (so that their components are identical with those found by local observers in their laboratory measurements) and to use the relation of these frames to the coordinate system only for studying the change of the quantities in the course of their propagation through the spacetime.

It is worth noting that a measure (element of volume, i.e., integration, which is needed to postulate variational principles for fields) on an n-dimensional manifold is determined by the choice of the unit n-form. A metric is thus not necessary, however, it is natural for a given metric to choose as the unit n-form the exterior product of some orthonormal dual basis (e.g., in the above example $r^2\cos\varphi d\varphi \wedge d\lambda = \theta^1 \wedge \theta^2$).

Exercise 6 *Prove that a tensor $F \in T^0_2$, which for all $X \in T$ gives $F(X,X) = 0$, must be antisymmetric, i.e., it must be a 2-form.*

Exercise 7 *Investigate the parallel transport (affine connection) on a general two-dimensional surface (and in a plane and on a sphere in particular) defined by a compass in vector H of a magnetic field with a general magnetic deviation.*

Exercise 8 *Calculate both in coordinate and orthonormalized bases of vectors the (torsion-free) metric connection and corresponding curvature of two-dimensional spacetime with the metric given by*

$$ds^2 = -f(x)^2dt^2 + dx^2 \ .$$

Discuss the results in connection with the problem of accelerated observers in special relativity.

Exercise 9 *Calculate affine connections and curvature in a two-dimensional conformally flat spacetime*

$$ds^2 = f^2(t,x)\left(-dt^2 + dx^2\right) \ .$$

Investigate the particular case $f = 1/(\cos(x+t)\cos(x-t))$, and explain the result.

Exercise 10 *For a general affine connection ∇ prove Theorem 4 that (i) the torsion Q and (ii) the curvature R defined by equations Eqs. (A.41) and (A.42) are tensors.*

Exercise 11 *Using the equations of structure (A.47) and (A.48) prove Theorem 6 that (i) for a metric connection (for which Eq. (A.50) is valid) the components of the tensor of curvature satisfy the antisymmetry (A.53), (ii) that for a torsion-free connection they satisfy the symmetry (A.54), and (iii) the Bianchi identities (A.55).*

3.2 Principles of general relativity

General relativity is the relativistic theory of gravitation. It is required from any new physical theory to satisfy the so-called **principle of correspondence**, i.e., to give the same results as the previous theories to be revised in the limiting cases when the assumptions of these theories are satisfied and they successfully explain the observed facts (cf. Newtonian mechanics as limit of special relativity for $c \to \infty$ or classical mechanics as the limit of quantum mechanics for $h \to 0$). Historically, the reason for revising the Newtonian theory of gravitation was not its contradiction with experimental results,[1] but its inconsistency with special relativity, which has to be accepted due to the experimental results in electrodynamics. The Newton law of gravitation (2.3) supposes instantaneous interaction at a distance. This is inconsistent with the relativity of simultaneity. The Newtonian theory of gravitation is often equivalently formulated as a field theory with a scalar gravitational potential Φ determining the gravitational force

$$\vec{F}_G = -m\vec{\nabla}\Phi \ , \tag{3.16}$$

and being determined by the mass M of a point-like source

$$\Phi = -G\frac{M}{r} \ , \tag{3.17}$$

or as a linear superposition of such fields corresponding to several sources. Φ can also be determined in differential form by the distribution of mass density ρ according to the Poisson equation

$$\Delta\Phi = 4\pi G\rho \ , \tag{3.18}$$

which is equivalent, because $\frac{1}{r}$ is the Green's function for the operator Δ.

This field formulation of Newton's gravity could, in principle, be adapted (e.g., replacing here the Laplacian operator $\Delta \equiv \nabla^2$ by the d'Alembertian $\Box \equiv -c^{-2}\partial_t^2 + \Delta$, i.e., integrating the retarded potential along the past light-cone) to carry the gravitational information with the speed of light (or some other velocity, but then a privileged 'rest frame' of the field must be postulated, which would violate the equivalence of all inertial systems).

However, there are additional features in Newton's gravitation which are taken into account in the formulation of the relativistic theory. First is the equivalence of inertial mass (m in Eq. (2.2) defined as the coefficient of proportion of the force acting on a body to the body's acceleration with respect to an inertial frame caused by this force) and the passive gravitational mass (m in Eq. (3.16) defining the force by which a given gravitational field acts on any body). Because there is also no known way of screening the gravitational interaction, all bodies (free of any other force) fall with the same acceleration in the same field, and there is no local experiment distinguishing the

[1] This is due to insufficient precision of previous measurements, which must be increased to test the tiny quantitative differences between predictions of Newtonian gravitation and general relativity within the solar system or to enable observations of distant objects in some of which the differences are large.

presence of a gravitational field from the noninertiality of the reference frame.[2] This experimental evidence has been tested starting from the historical experiment of Galileo in Pisa up to the present cosmic experiments. This (up to now confirmed) experimental fact is known as the **principle of equivalence**, which has several alternative formulations differing in their strength. A weak formulation of the equivalence principle says that gravity is a kinematic force, i.e., it can be described as a non-inertiality of the reference frame, and all bodies free of any other external interaction (with identical initial conditions) will follow in the same gravitational field the same worldline independent of their inner structure.[3] A strong formulation of the equivalence principle says that any local physical experiments have the same outcome (i.e., the physical laws are the same) in any locally inertial (i.e., freely falling) reference frame.[4]

If we accept the equivalence of the gravitational field and the noninertiality of the reference frame, we see (e.g., from the example shown by Eq. (2.20)) that the gravitational field must be described by the components of the metric tensor which give the relation between the coordinates and measurements of space and time distances. The physical meaning of these components can be understood from the following experiment. Imagine a source of monochromatic light with a frequency ν placed in a homogeneous (and stationary) gravitational field with gravitational acceleration g. Let an absorber of the light be placed on a different gravitational potential $\Phi_{abs} = \Phi_{em} + lg$ (l being their vertical distance; imagine for the sake of simplicity the absorber to be just above the emitter). Because of the stationarity of the problem, the absorber will register the same number of waves in the same (synchronized) interval of coordinate time. The frequency ν of the emitted photons must remain constant with respect to the inertial frame which was at rest at the instant of emission, i.e., to the reference frame which just started to fall with an acceleration g at that moment. The photon will reach the absorber after the time $t = l/c$ when the inertial frame will move with the velocity $v = gt = (\Phi_{abs} - \Phi_{em})/c$. The absorber is thus receding with respect to this inertial frame and will see the photon (gravitationally) red-shifted for $\Delta\nu/\nu \sim \Delta\Phi/c^2$.[5] It means that the proper time τ of the absorber is running with a higher rate than the coordinate time just in the ratio

$$d\tau/dt = \sqrt{|g_{00}|} \simeq 1 + \Phi/c^2 \ , \tag{3.19}$$

if the coordinate time is chosen to be the proper time of an observer who is at rest at infinity, where $\Phi = 0$. This is consistent with the relation between the metric of the accelerated observer and the potential of his apparent gravitational field – see Eq. (2.20)). The components of the metric tensor have thus the meaning of gravitational potentials, their derivatives (the components of 1-forms of connections) the meaning of gravitational acceleration, and their second derivatives meaning of gradients of intensity of the gravitational field, i.e., they specify the tidal forces.

Another idea which played an important heuristic role in the development of general relativity was **Mach's principle**, according to which the preference of the inertial frames (in the Newtonian mechanics) must be due to the interaction of local bodies with the matter in distant parts of the universe. This idea, which has not been convincingly mathematically incorporated into

[2]A non-local experiment may consist in ranging the position of a test particle with respect to another one far away from the gravitating body, where the gravity vanishes, or at least with respect to a slightly distant particle, where the gravitational field has different intensity and direction.

[3]The distribution of mass in bodies extending on a scale comparable to inhomogeneities of the gravitational field is not in this sense treated as their inner structure but rather as several mutually interacting bodies in different fields. Several mechanisms based on this principle have been proposed to change orbits of satellites or the spin-orbital interaction in binaries is another example of this mechanism.

[4]Speaking about 'the same shape' of physical laws in different frames, the number of dynamical variables must be specified. Obviously, by adding new parameters, any law can be generalized to any wider class of frames.

[5]This experiment has been performed in practice historically first by measurement the gravitational redshift of spectral lines in the solar atmosphere and recently also on the Earth using the Mössbauer effect (with l of the order of meters).

the Newton's theory, is confirmed by the relativistic effect of the dragging of inertial frames (cf. exercise 14).

3.3 Motion in curved spacetime

In general relativity we suppose that the spacetime is a 4-dimensional pseudo-Riemannian[6] manifold with the signature $(-+++)$, i.e., it locally resembles the Minkowskian spacetime and an orthonormal frame of basis vectors can locally correspond to the inertial systems in the Newtonian physics. The metric torsion-less connection (i.e., the Riemannian connection) determines how the directions of inertial coordinate axes should be prolonged to the neighbouring events. It means also that the metric connection determines how the 4-momentum of free particles (proportional to their 4-velocities $U = \frac{d}{d\tau}$ and thus being tangent to their worldlines) should be transported along the worldlines

$$0 = \frac{D}{d\tau}U = \left(\frac{d}{d\tau}U^\iota + U^\kappa {\omega^\iota}_\kappa(\frac{d}{d\tau})\right) e_\kappa \ , \tag{3.20}$$

so that they are thus 'straight'. An alternative view on how a particle can recognize its proper way in curved spacetime is yielded by Feynman's approach to quantum mechanics, using the integrals on paths. The wave-function is a superposition of contributions of all paths

$$\psi = \int \exp \frac{iS}{h} \ , \tag{3.21}$$

but, due to the rapidly varying phases, only the contributions of paths close to the extremal of action S are increased by their interference, while others mutually cancel. In this view, free particles in spacetime (the action of which is given, in agreement with Eq. (A.56), by their proper time) spread along the geodesics, i.e., the extremal line.[7] Following Theorem 7 the resulting equation of motion again is (3.20).

If we follow different geodesic worldlines $x = x(\tau, \epsilon)$ labeled by a 'comoving' coordinate ϵ, each one parametrized by its proper time τ, the 4-velocity $U = \frac{\partial}{\partial \tau}$ commutes with the vector $\xi = \frac{\partial}{\partial \epsilon}$ of displacement of the geodesic (because they are both coordinate basis-vectors in the coordinates τ, ϵ). Hence in the torsion-free connection

$$0 = [U, \xi] = \nabla_U \xi - \nabla_\xi U \ , \tag{3.22}$$

where $v = \nabla_U \xi$ is the relative velocity of the worldlines. Their relative acceleration thus is

$$a = \nabla_U v = \nabla_U \nabla_U \xi = \nabla_U \nabla_\xi U = ([\nabla_U, \nabla_\xi] - \nabla_{[U,\xi]})U = \langle R, U \otimes U \otimes \xi \rangle \tag{3.23}$$

(because the terms added to correspond to the definition (A.42) are zero due to (3.22) and the equation of geodesic motion $\nabla_U U = 0$). According to this equation of geodesic deviation the relative acceleration of nearby geodesic worldlines thus is determined by the curvature of spacetime. This corresponds to the relative acceleration of freely falling particles due to the tidal forces in Newtonian theory, $\nabla a = -\nabla \nabla \Phi$.

[6]Following the Theorem 6 we mean by Riemannian geometry the torsion-free metric connection and the corresponding curvature. By 'pseudo-' we distinguish the case of the indefinit metric.

[7]For the time-like lines it is the longest line, because the proper time of an accelerated particle with relativistic velocities will be smaller like in the twin-paradox.

3.4 Field equations

If we already know that the spacetime metric (or its equivalents in non-coordinate vector bases) plays the role of gravitational potentials and that it forces (by the components of 1-forms of connection) the particles to move along its geodesics, we need to find the field equations to complete the theory. These equations must be a generalization of Eq. (3.18) and they must determine the influence of the sources (bodies) on the metric. In Newton's theory the source of the gravitation is the active gravitational mass, which is postulated to be equal also to the passive one and to the inertial mass. This is necessary to satisfy the conservation of momentum of an isolated self-gravitating system. Because in relativity the inertial mass is given by the total energy of the bodies (including their kinetic and even the bounding potential energy), the source term in the field equations cannot be given simply as a sum of charges of individual particles (e.g., their rest masses). However, the density of energy is not a scalar quantity, but only one of the components of the stress-energy tensor. The value of the energy density with respect to a particular reference frame of an observer is determined by the projection of the stress-energy tensor into the direction of his 4-velocity. It is thus reasonable to suppose that the whole stress-energy tensor $T_{\mu\nu}$ will be the source term in a frame-independent formulation of the field equations.[8]

The left-hand side of the field equations can be constructed, in principle, in an infinite number of ways. Only some of them will lead to a reasonable theory yielding, in accordance with the principle of correspondence, the Newtonian approximation as their special case. There have been found several different generalizations predicting slightly or more different post-Newtonian corrections. The choice between them is, first of all, a subject of their experimental verification. Einstein's standard general relativity has been verified to within the observational limits, and this theory is remarkable amongst others by its simplicity. Experimentally, there cannot be excluded also those generalizations, which can for a special choice of their free parameters explain the same results as the Einstein's theory. However, it is thus practical to prefer just this simplest theory. Several alternative ways of reasoning can be used to derive the equations of Einstein's theory and to demonstrate its predominance consisting in its mathematical beauty. Nevertheless, it is important to remember that, as with any other physical theory, the crucial decision is on the nature, and our aesthetic criteria would be modified to understand and justify the theory approved by experience. We will thus not make as though the equations here are derived in a unique possible way. Instead, we will postulate it (with only a few explanatory remarks) and we will show how these equations give the Newtonian approximation.

If we suppose that the components of the metric tensor should substitute the potential in Eq. (3.18), it is worth noting that the equation is no longer linear in these variables, because the (contravariant, i.e., inverse) metric is implicitly included also in the differential operator ($\Delta = g^{ij}\nabla_i\nabla_j$). If we suppose the field equations to be also differential of the second order, we can look for equations linear in the second derivatives, but nonlinear in the first derivatives and the values of the components. Because the stress-energy tensor (which is of the type (0,2)) is supposed to determine the source term, we can try simply to equate it to a tensor of the same order constructed from the geometry (despite that this is not the only possible construction). A tensor which can be constructed from the metric as linear in its second derivatives is the Riemann tensor $R^{\iota}{}_{\kappa\lambda\mu}$, i.e., the tensor of curvature of the torsion-free metric connection. This tensor is of type (0,4), however, its contraction (e.g., in the first and the third index, which is named the Ricci tensor $R_{\kappa\mu} = R^{\iota}{}_{\kappa\iota\mu}$) is just of the type (0,2) as is the stress-energy tensor. The contraction of the Ricci tensor is the scalar curvature, which is also linear in the second derivatives, so that multiplied by the metric itself it is also of the desired type. There can also be a term proportional to the metric only (not

[8]Just the mixed space- and time- components T_{0k}, representing the momentum density of the matter can influence the solution in terms of Mach's principle.

to its derivatives). Such a term (named for its possible significance the cosmological term) can also be treated as the stress-energy tensor of the vacuum and added to the right-hand side; we will thus omit it for now. The expected form of the field equations thus is

$$R_{\iota\kappa} + aRg_{\iota\kappa} = kT_{\iota\kappa} \ . \tag{3.24}$$

The constant a can be determined by the following reasoning. The Riemann tensor identically satisfies the Bianchi identities

$$R^{\iota}{}_{\kappa\lambda\mu;\nu} + R^{\iota}{}_{\kappa\nu\lambda;\mu} + R^{\iota}{}_{\kappa\mu\nu;\lambda} = 0 \ , \tag{3.25}$$

(see Theorem 6) which, contracted e.g., in pairs of indices ι, λ and $\kappa\mu$, yield

$$R_{,\nu} - 2R^{\mu}{}_{\nu;\mu} = 0 \ . \tag{3.26}$$

Consequently, if we choose $a = \frac{1}{2}$, the left-hand side of the field equations (3.24) will be given by the so-called Einstein tensor

$$G_{\iota\kappa} = R_{\iota\kappa} - \frac{1}{2}Rg_{\iota\kappa} \ , \tag{3.27}$$

the divergence of which vanishes identically. The conservation of the 4-momentum of the mass

$$0 = T^{\iota\kappa}{}_{;\kappa} \tag{3.28}$$

(i.e., of the non-gravitational fields on the right-hand side) which, following Noether's[9] theorem, should be consequence of a symmetry of the spacetime, will thus follow from the field equations determining the spacetime geometry.

The constant k, which should be proportional to the gravitational constant, must be found according to the principle of correspondence from the weak-field approximation. This approximation must also verify if the field equations in this form give a viable generalization of Newton's theory of gravity. Let us thus bring first the final result, which will then be verified in the next Section. The Einstein's field equations read

$$G_{\iota\kappa} \equiv R_{\iota\kappa} - \frac{1}{2}Rg_{\iota\kappa} = \frac{8\pi G}{c^4}T_{\iota\kappa} \ . \tag{3.29}$$

Exercise 12 *Using the symmetries of the Riemann tensor*

$$R_{(ik)lm} = R_{ik(lm)} = R_{i(klm)} = 0 \ ,$$

prove the symmetry of the Ricci tensor.

Exercise 13 *Using the symmetries of the Riemann tensor*

$$R_{(ik)lm} = R_{ik(lm)} = R_{i(klm)} = 0 \ ,$$

show that the number of its algebraically independent components in an n-dimensional manifold is $n^2(n^2-1)/12$.

[9]According to Noether's theorem, each law of conservation corresponds to a symmetry of the problem in question.

3.5 Weak-field approximation

Let us suppose that in the coordinate frame dx^ι of nearly Minkowskian coordinates (we now choose $x^0 = ct$) the spacetime metric is described by a small smooth perturbation $\gamma_{\iota\kappa}$ of the Minkowskian metric $\eta_{\iota\kappa} = \mathrm{diag}(-1, +1, +1, +1)$,

$$g_{\iota\kappa} = \eta_{\iota\kappa} + \gamma_{\iota\kappa} \; . \tag{3.30}$$

By the smallness we understand that in a Taylor's expansion of any function of the metric, the higher order terms can be neglected, and by the smoothness that also the derivatives of the perturbation are negligible of the same order. The contravariant, i.e., the inverse metric[10] thus reads

$$(g^{-1})^{\iota\kappa} = \eta^{\iota\kappa} - \gamma^{\iota\kappa} \; , \tag{3.31}$$

where the raising (or lowering) of indices is performed by η. For the metric connection

$$d\gamma_{\iota\kappa} = \omega_{\iota\kappa} + \omega_{\kappa\iota} \; , \tag{3.32}$$

in the coordinate frame, the first equations of structure

$$0 = \omega_{\iota\kappa} \wedge dx^\kappa \tag{3.33}$$

have solutions (cf. Eq. (A.52))

$$\omega_{\iota\kappa} = \frac{1}{2}\left[d\gamma_{\iota\kappa} + (\gamma_{\lambda\iota,\kappa} - \gamma_{\kappa\lambda,\iota})dx^\lambda\right] \; . \tag{3.34}$$

In the second equations of structure the quadratic term can be neglected, hence

$$\Omega_{\iota\kappa} = d\omega_{\iota\kappa} = \frac{1}{2}(\gamma_{\lambda\iota,\kappa\mu} - \gamma_{\kappa\lambda,\iota\mu})dx^\mu \wedge dx^\lambda \; . \tag{3.35}$$

The Ricci tensor can be obtained by contraction with $\frac{\partial}{\partial x^\iota}$ in the form

$$R_{\kappa\lambda} = \frac{1}{2}\left(\gamma_{\lambda\iota,\kappa}{}^{\iota} + \gamma_{\kappa\iota,\lambda}{}^{\iota} - \gamma^{\iota}{}_{\iota,\kappa\lambda} - \gamma_{\kappa\lambda,\iota}{}^{\iota}\right) \; , \tag{3.36}$$

and the scalar curvature is

$$R = \gamma_{\mu\iota,}{}^{\mu\iota} - \gamma^{\mu}{}_{\mu,}{}^{\iota}{}_{\iota} \; . \tag{3.37}$$

The Einstein tensor thus reads

$$\begin{aligned} G_{\kappa\lambda} &= \frac{1}{2}\left(\gamma_{\lambda\iota,\kappa}{}^{\iota} + \gamma_{\kappa\iota,\lambda}{}^{\iota} - \gamma^{\iota}{}_{\iota,\kappa\lambda} - \gamma_{\kappa\lambda,\iota}{}^{\iota} - \eta_{\kappa\lambda}\gamma_{\mu\iota,}{}^{\mu\iota} + \eta_{\kappa\lambda}\gamma^{\mu}{}_{\mu,}{}^{\iota}{}_{\iota}\right) = \\ &= \frac{1}{2}\left(\bar{\gamma}_{\lambda\iota,\kappa}{}^{\iota} + \bar{\gamma}_{\kappa\iota,\lambda}{}^{\iota} - \bar{\gamma}_{\kappa\lambda,\iota}{}^{\iota} - \eta_{\kappa\lambda}\bar{\gamma}_{\mu\iota,}{}^{\mu\iota}\right) \; , \end{aligned} \tag{3.38}$$

where the so-called trace-reversed perturbation reads

$$\bar{\gamma}_{\iota\kappa} = \gamma_{\iota\kappa} - \frac{1}{2}\eta_{\iota\kappa}\gamma^{\lambda}{}_{\lambda} \; . \tag{3.39}$$

[10] Note that the exponent -1 is usually omitted only for brevity, when no confusion can arise in the presence of only one metric. However, according to Definition 5 the covariant and contravariant metric tensors are really inverse functions and should be distinguished.

Up to now we hold the coordinate system and its coordinate basis as given. However, physically the same geometry

$$ds^2 = g_{\iota\kappa} dx^\iota dx^\kappa = \tilde{g}_{\iota\kappa} d\tilde{x}^\iota d\tilde{x}^\kappa \simeq \tilde{g}_{\iota\kappa}(dx^\iota dx^\kappa + d\xi^\iota dx^\kappa + dx^\iota d\xi^\kappa) \tag{3.40}$$

is described by different metric tensors

$$\tilde{g}_{\iota\kappa} = g_{\iota\kappa} - \frac{\partial \xi^\lambda}{\partial x^\iota} g_{\lambda\kappa} - g_{\iota\lambda} \frac{\partial \xi^\lambda}{\partial x^\kappa} \ , \tag{3.41}$$

if the coordinates (and the vector bases) are pertubed

$$d\tilde{x}^\iota = dx^\iota + d\xi^\iota(x) \ . \tag{3.42}$$

For the same order of magnitude of the coordinate perturbation and the perturbation of the Minkowskian metric

$$\tilde{\gamma}_{\iota\kappa} = \gamma_{\iota\kappa} - \xi_{\kappa,\iota} - \xi_{\iota,\kappa} \ , \tag{3.43}$$

and

$$\tilde{\bar{\gamma}}_{\iota\kappa} = \bar{\gamma}_{\iota\kappa} - \xi_{\kappa,\iota} - \xi_{\iota,\kappa} + \eta_{\iota\kappa} \xi^\lambda{}_{,\lambda} \ . \tag{3.44}$$

The divergence of this expression reads

$$\tilde{\bar{\gamma}}^\iota{}_{\kappa,\iota} = \bar{\gamma}^\iota{}_{\kappa,\iota} - \xi_{\kappa,}{}^\iota{}_\iota \ . \tag{3.45}$$

For a known divergence $\bar{\gamma}^\iota{}_{\kappa,\iota}$ it is thus possible to solve this Laplace equations for ξ_κ to get the divergence on the left-hand side vanishing. In this so-called Lorentz gauge all but the third term in the Einstein tensor (3.38) vanish, and it becomes

$$G_{\kappa\lambda} = -\frac{1}{2} \bar{\gamma}_{\kappa\lambda,\iota}{}^\iota = -\frac{1}{2} \Box \bar{\gamma}_{\kappa\lambda} \ , \tag{3.46}$$

where we have already omitted the mark ˜ of the new coordinate system, however, we suppose implicitely that our choice of the coordinate system ensures the condition

$$\bar{\gamma}^\iota{}_{\kappa,\iota} = 0 \ . \tag{3.47}$$

The Newtonian approximation should correspond to Eq. (3.18) in the case of a static distribution of the graviting mass. For coherent dust, the stress-energy tensor reads

$$T^{\iota\kappa} = \rho U^\iota U^\kappa \ , \tag{3.48}$$

hence in the comoving coordinate system the stress-energy tensor has the only non-zero component $T^{00} = \rho c^2$. Supposing the field equations in the form

$$-\frac{1}{2} \Box \bar{\gamma}_{\iota\kappa} \simeq G_{\iota\kappa} = k T_{\iota\kappa} \ , \tag{3.49}$$

we thus find that the only non-trivial component of $\bar{\gamma}_{\iota\kappa}$ (i.e., which is not a solution of the homogeneous, i.e., vacuum equations) is $\bar{\gamma}_{00}$. Consequently,

$$\gamma_{00} = \bar{\gamma}_{00} - \frac{1}{2} \eta_{00} \bar{\gamma}^\iota{}_\iota = \frac{1}{2} \bar{\gamma}_{00} \ , \qquad \gamma_{ii} = -\frac{1}{2} \eta_{ii} \bar{\gamma}^\iota{}_\iota = \frac{1}{2} \bar{\gamma}_{00} \ . \tag{3.50}$$

Following Eq. (3.19), the component γ_{00} should correspond to the gravitational potential ($g_{00} = -1 + \gamma_{00} = -1 - 2\Phi/c^2$) and in the static case it obeys the equation

$$\frac{2}{c^2}\Delta\Phi = -\Delta\gamma_{00} = -\frac{1}{2}\Delta\bar{\gamma}_{00} = kT_{00} = kc^2\rho \; , \tag{3.51}$$

which is identical with Newton's one just if $k = \frac{8\pi G}{c^4}$ (or 8π in the geometric units $c = 1 = G$) as it was supposed in Eq. (3.29).

Note that in addition to the predicted slowing down of the time in the gravitational field by the term γ_{00}, there is perturbed also the geometry of the space by the terms γ_{ii}. These terms have negligible influence on the motion much slower than the speed of light, but they are comparable for relativistic velocities. This is why the relativistic bending of light predicted by Einstein is twice higher than that predicted by Soldner on the basis of the corpuscular theory of light and Newtonian gravity.

The Poisson equation (3.51) for γ_{00} (or Φ) can be integrated using the corresponding Green's function (cf. Eq. (3.17)). It can be done also with the general weak-field equations

$$-\frac{1}{2}\Box\bar{\gamma}_{\iota\kappa} = \frac{8\pi G}{c^4}T_{\iota\kappa} \tag{3.52}$$

valid also for the non-stationary and non-static case (i.e., if there are also non-zero time derivatives on the left-hand side and other components of T on the right-hand side). The corresponding Green's function is given, in analogy with the classical electrodynamics, by the retarded potentials, i.e., we have to integrate along the past light-cone. Unlike Newton theory, in the weak-field limit of general relativity, the momentum or flow of energy T^{0i} and the stress T^{ij} produce the gravity (in a similar way as in electrodynamics where the currents induce the magnetic field in addition to the electric field produced by charges in electrostatics).

Exercise 14 *Calculate in the weak-field approximation (i.e., according to Eqs. (3.30), (3.39) and (3.49)) the metric inside a rotating homogeneous mass shell and investigate its influence on motion of test particles (cf. Problem 3 on p. 89 of [6]).*

Exercise 15 *Show that in linearized theory there is no gravitational interaction between two parallel beams of photons.*

3.6 Gravitational radiation

3.6.1 Newtonian preliminaries

In the Newtonian theory of gravitation a point-like body of mass m generates a scalar gravitational potential

$$\varphi(r) = -\frac{Gm}{r} \; . \tag{3.53}$$

If we split the mass like a dumbbell into two equal parts, each one shifted for a distance x in opposite directions, then we will get a deeper total potential in this direction

$$\varphi(r) = -\frac{Gm}{2(r-x)} - \frac{Gm}{2(r+x)} = -\frac{Gmr}{r^2 - x^2} \simeq -\frac{Gm}{r}\left(1 + \frac{x^2}{r^2}\right) \; , \tag{3.54}$$

and a shallower one in the perpendicular directions

$$\varphi(r) = -\frac{Gm}{\sqrt{r^2 + x^2}} \simeq -\frac{Gm}{r}\left(1 - \frac{x^2}{2r^2}\right) \; . \tag{3.55}$$

The perturbation

$$\delta\varphi(r) \simeq \frac{Gmx^2}{r^3} , \tag{3.56}$$

is the quadrupole term in the multipole expansion of the gravitational field of an isolated graviting system. In principle, we can thus construct a gravitational telegraph, if we will measure by some extremely sensitive torsion weight the change of gravitational acceleration

$$\delta g(r) = \nabla\delta\varphi \simeq \nabla\frac{Gmx^2}{r^3} \simeq \frac{Gmx^2}{r^4} , \tag{3.57}$$

caused by turning the source-dumbbell along or perpendicularly to the line of sight of the detector. To avoid the necessity of having fixed the distance r of the detector, we can also use a gradientometer, which will measure the change of relative accelerations of two test masses separated for a distance 2ξ

$$\delta g(r+\xi) - \delta g(r-\xi) = 2\xi\frac{d^2}{dr^2}\delta\varphi \simeq \xi\frac{Gmx^2}{r^5} . \tag{3.58}$$

The information about the change of orientation of the source is carried by some energy, which can be extracted from the change of gravitational field, e.g. if we let fall down a test mass μ in the phase of stronger gravitational field and we rise it up again in the weaker gravitational field. Or, to solve the need of exchange of momentum of the oscillating body, we can use again the two masses ($\frac{\mu}{2}$ each) in distance 2ξ as the absorber and turn them properly with the phase of variations of the gravitational field. The amount of energy absorbed by one absorber in one cycle of the field variation is thus of the order

$$\delta E = \frac{\mu}{2}\delta(2\varphi(r) - \varphi(r+\xi) - \varphi g(r-\xi)) \simeq \mu\xi^2\frac{d^2}{dr^2}\delta\varphi \simeq \frac{Gmx^2\mu\xi^2}{r^5} . \tag{3.59}$$

There can be many absorbers in such a gravitational energy transmitter, however, their own quadrupole moments will screen the variations of the field of source if they will be of the same order. The maximum energy which can be lost in one cycle by the source due to its interaction with the absorbers is thus of the order

$$\Delta E \simeq \frac{Gm^2x^4}{r^5} . \tag{3.60}$$

The rate of energy loses in such processes[11] is obviously dependent on the distribution of the absorbers in space. Their interaction with the source is instantaneous according to the Newtonian theory. This contradicts to the special relativity, according to which the signal can propagate from the source to the absorbers and back with the speed c of light at maximum. Consequently, in one period t, the source must give to the gravitational field its energy of the order of

$$\frac{dE}{dt} \simeq \frac{G}{c^5}\left(\frac{mx^2}{t^3}\right)^2 \tag{3.61}$$

for possible absorbers at distances $r \simeq ct$ and behind, without knowing if they are present and in what phase. This energy is radiated away from the source in the form of perturbations of the gravitational field, even if there are no absorbers in distances much larger than ct. The exact amount of this energy is given by the dynamics of the space-time variability of the gravitational field as well as on the number of degrees of its freedom, which is different in the scalar Newtonian theory and in the tensor general relativity or in some alternative theories.[12]

[11] Such processes are observed as different effects of gravitational resonances e.g. in Solar system.

[12] The Newtonian analog of general-relativistic gravitational waves gives a warning concerning the complexity of the problem of interpretation of possibly observed gravitational waves. The observed loose of source energy can be

3.6.2 Weak-field solutions in vacuum

The general relativistic equation (3.52) for weak gravitational field reduces in vacuum to

$$\Box \bar{\gamma}_{\iota\kappa} = 0 \ , \tag{3.62}$$

with the gauge condition (3.47). General solution of this homogeneous equation can be written as a superposition of Fourier modes

$$\bar{\gamma}_{\iota\kappa}(x) = h_{\iota\kappa} \exp(i k_\lambda x^\lambda) \ , \tag{3.63}$$

satisfying, according to (3.62), the condition

$$k_\lambda k^\lambda = 0 \ , \tag{3.64}$$

i.e., the wave-vector k must be on the light-cone, and according to the gauge condition (3.47)

$$h^{\iota\kappa} k_\iota = 0 \ , \tag{3.65}$$

i.e., the tensor h of the wave amplitude must be orthogonal to the wave-vector. This equation imposes four restrictions on otherwise 10 algebraically independent components of h. However, the gauge transformation (3.42) is not defined uniquely by the condition of vanishing of the expression (3.45), because any additional transformation generated by vector ξ which satisfies condition

$$\Box \xi_\kappa = 0 \ , \tag{3.66}$$

will not change the divergence of $\bar{\gamma}$ (i.e. the orthogonality of h with respect to k). In particular, for a planar wave (3.63) we can choose ξ satisfying

$$\xi^\iota{}_{,\iota} = -\frac{1}{2}\bar{\gamma}^\iota{}_\iota \ , \tag{3.67}$$

so that following (3.45) the new tensor $\bar{\gamma}$ will be traceless,

$$\bar{\gamma}^\iota{}_\iota = -\gamma^\iota{}_\iota = 0 \ , \tag{3.68}$$

and hence $\bar{\gamma} \equiv \gamma$. Obviously, ξ must be chosen also as a planar wave with the same wave-vector

$$\xi_\kappa = X_\kappa \exp(i k_\lambda x^\lambda) \ , \tag{3.69}$$

and with amplitude satisfying

$$X^\iota k_\iota = \frac{i}{2} h^\iota{}_\iota \ . \tag{3.70}$$

The other three still free components of X can be used to suppress all but the two transverse components of h.

influenced by its interaction with some absorbers in the near zone or with some inciding gravitational radiation and a detectable intensity of gravitational waves can be dumped by some absorbers between the source and the detector.