# Chapter 2

# Special relativity

In this chapter (which is supplementary to chapter 1 of [6]) will be discussed basic concepts of spacetime in special relativity and their consequences for general relativity, which can be deduced from the relativistic motion of accelerated observers. There is given only a short review of special-relativistic dynamics and electrodynamics. However, it is recommended to the students to repeat these disciplines in details in context of problems discussed here.

## 2.1 Spacetime

The Newtonian mechanics is invariant with respect to Galileo's group of transformations, i.e., its dynamical laws have the same form in any inertial frame. Let us demonstrate this fact on one-dimensional motions of point masses drawn in the space-time diagram, i.e., in a representation of the **spacetime**, the set of **events**, each one characterized by its position in space and time. On the left panel of Fig. 2.1, drawn with respect to an inertial frame $S$ with coordinates $\{t, x\}$ the **worldline**, i.e., the subset of the spacetime corresponding to the events of occurrence of a particular body, of a reference point of the beginning of another inertial frame $S'$ is a straight line $x = vt$. To transform this diagram to $S'$, i.e.,

$$x' = x - vt \ , \qquad t' = t \ , \tag{2.1}$$

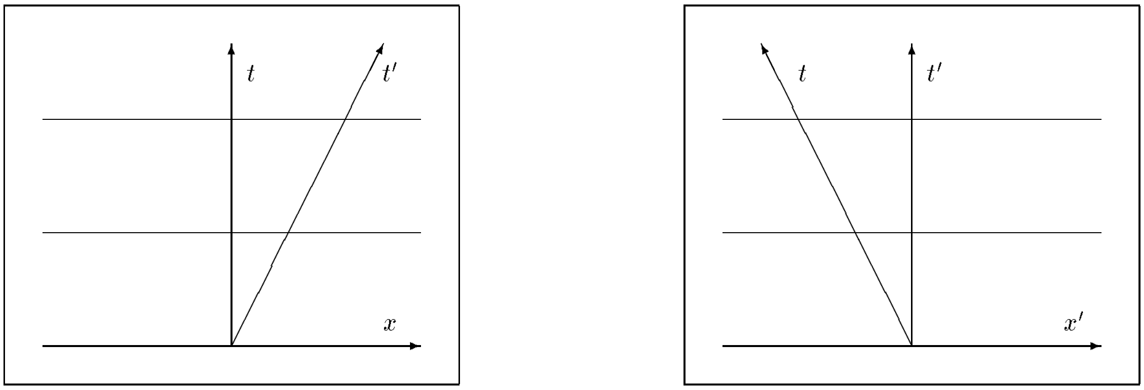


Figure 2.1: Galileo's transformation.

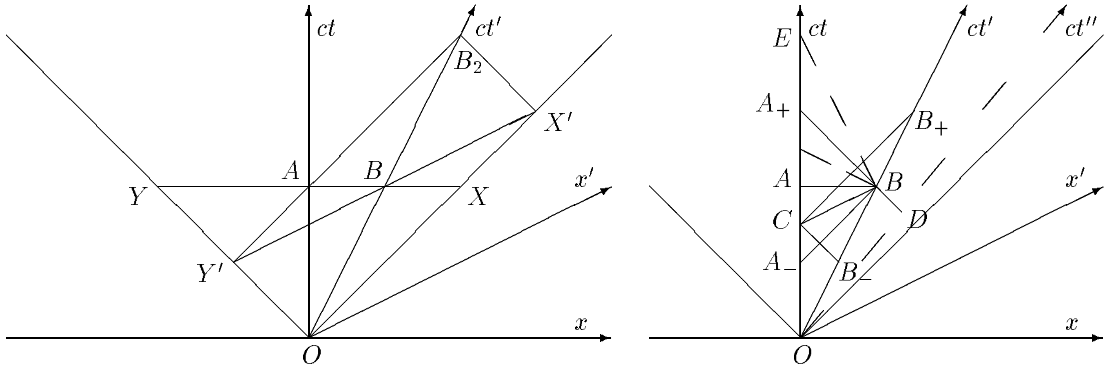


Figure 2.2: Relativity of simultaneity.

means to skew it by shifting its space slices (see the right panel). Note, that the Cartesian coordinates $t, \vec{x}$ determine both the topology of the spacetime as well as the metric properties of both space and time (i.e., the spatial distance $\Delta s$ is given by $\Delta s^2 = \sum_{i=1}^{3} \Delta x^i \Delta x^i$ and the temporal distance by $\Delta t$). Because the force $\vec{F}$ and inertial mass $m$ are supposed to be invariant with respect to Galileo's transformation (2.1), also the Newton's law

$$m \frac{d^2}{dt'^2} \vec{x}' = m \frac{d^2}{dt^2} \vec{x} = \vec{F} \tag{2.2}$$

is invariant. An example of a force (and its invariance) is the Newtonian gravitational force

$$\vec{F}_G = -G \frac{mM}{r^2} \vec{r}^0 \ . \tag{2.3}$$

Unlike Newtonian mechanics, classical electrodynamics is not invariant with respect to the Galileo's group of transformations (2.1). It can be seen e.g., from the fact that while two charges at rest interact by the electrostatic Coulomb force only, in an inertial frame $S'$ each charge represents also an electric current inducing the magnetic field which exerts an additional magnetic component of the Lorentz force on the other charge. Obviously, also the Maxwell's equations are not invariant and, in particular their consequence of the speed of light equal to $c$ could be valid only in an privileged 'rest frame of the ether' according to the Newtonian point of view. However, it has been verified experimentally by Michelson interferometer that the speed of light is the same in all inertial systems. It means that if two light signals will be radiated forward and backward from the event $O$ (see Fig. 2.2, left) then, at the time $t_A$ of the static observer, they will be found in equal distances $AX = AY$. However, another inertial observer at event $B$ can find equal distances to the worldlines of both light signals only if he measures them in the direction $X'Y'$ oblique with respect to the space $XY$ of events simultaneous according to the static observer. It means that each inertial observer finds a different cross-section of the spacetime to be his proper space, i.e., the set of simultaneous events. The simultaneity can be found e.g., by synchronization using light-ranging. On Fig. 2.2, left, the event $B$ is simultaneous with events $X'$ and $Y'$ according to the observer $S'$, because it is in the middle of the time-interval $OB_2$ (spent by light to travel from $O$ to $X'$ or $Y'$ and back to the event $B_2$ on the worldline of the observer $S'$). Analogously on Fig. 2.2, right, $B$ is simultaneous with $A$ according to $S$ (because $A$ is in the middle between $A_-$ and $A_+$) but $B$ is simultaneous with $C$ according to $S'$ (because $B$ is in the middle between $B_-$ and $B_+$). Obviously, the obliquity of the proper space $X'Y'$ of the moving observer must be such to form

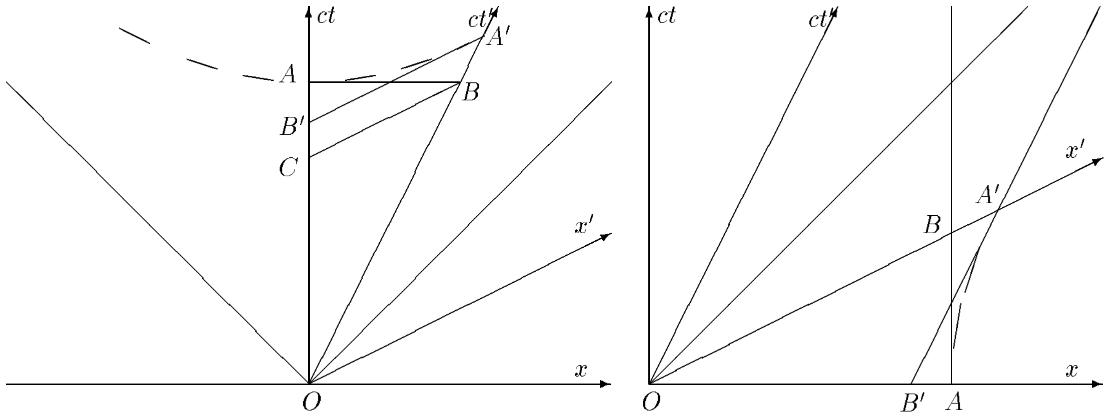


Figure 2.3: Relativity of spacetime units – dilatation of time (left) and contraction of lengths (right).

the other diagonal in the parallelogram $OX'B_2Y'$, i.e., by the same angle $\alpha \equiv \angle AOB = \angle XBX'$, given by relation $\tan\alpha = \beta \equiv \frac{v}{c}$, if the scales of axes $x$ and $t$ are chosen to give the slope of the worldline of photons equal to 45° (i.e., if the length $ct$ is plotted on the $t$-axis and $OX'B_2Y'$ is thus a rectangle).[1] Consequently, the time-sequence of two events will be found the same by all inertial observers only if one of them is inside the (future or past) light-cone of the other event and they can thus be joined by a (subluminal) worldline. In the opposite case of a couple of events outside the light-cone their sequence or simultaneity is relative.

## 2.2 Lorentz transformation

If the simultaneity is relative, then also the length of time-unit must be relative: if the interval $OA$ is chosen to be unity according to observer $S$, and if observer $S'$ would choose $OB$ to comply with $S$, then $S'$ would find $C$ simultaneous with $B$ and thus the clock of $S$ slower than his own (cf. Fig. 2.3 – left). To preserve the relativity of both observers an interval $OA'$ must be attributed as time-unit of $S'$ in such a way to satisfy $\frac{OA'}{OB} = \frac{OA}{OB'} \equiv \gamma$. Consequently,

$$\gamma^2 = \frac{OA'}{OB}\frac{OA}{OB'} = \frac{OB'}{OC}\frac{OA}{OB'} = \frac{OA}{OC} = \frac{OA}{OA - AC} = \frac{1}{1-\beta^2}\,. \tag{2.4}$$

The clocks of other observers thus seem to each of them to be slower than his own just in the ratio $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ (this is the so-called time dilatation). The set of all points $A'$ defining the unit of time

[1] In relativity there is often used a so-called 'geometric' system of units, in which the units of time and mass are chosen to give the speed of light and the gravitational constant $G = 1 = c$. This choice makes many calculations simpler and the result can always be expressed again in the standard units. However, there are also good reasons not to do it in a textbook. The first one is to demonstrate that relativity naturally works also for nonrelativistic problems. Nonrelativistic physicists use to measure the spacetime quantities in projection to the frame-vectors connected with their laboratories. In these frames the time-vector $\partial_t$ is usually chosen much longer than space-vectors $\partial_x$ to fit the usual slowness of daily life. The 'generality' of the relativistic approach enables also this choice. Another reason to prefer here the use of frames which are only orhogonal instead of the more symmetric orthonormal ones, is to make obvious the geometric difference between these vectors and their covectors $dt$, $dx$. Finally, like in the computer physics, the use of real units instead of those fitting the problem in question by numbers of order of 1 yields a natural check of reasonability of the results.

for observers moving from the event $O$ with different velocities forms the hyperbola $c^2t^2 - x^2 = 1$ (or actually a rotational hyperboloid $c^2t^2 - x^2 - y^2 - z^2 = 1$) with asymptotes on the light cone.

To keep the speed of light constant for all observers, the length of unit of spatial distances must be amplified in the same ratio $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. It is well known, that unlike the time dilatation, the length undergoes the relativistic contraction. However, this apparent asymmetry of space and time is caused by the more tricky way in which we compare the lengths of unit-rods of different observers (cf. Fig. 2.3 – right). If the world line of one end of the unit-rod of the unprimed observer coincides with the axis $t$, then the world line $AB$ of its other end will intersect the axis $x'$ at the event $B$. The primed observer will thus find its length $OB$ $\gamma$-times shorter than the length $OA'$ of his own unit-rod. On the other hand, the world lines $t'$ and $B'A'$ of the primed unit-rod intersect the axis $x$ in events $O$ and $B'$, so that also the unprimed observer will find it $\gamma$-times shorter than $OA$. The events in proper distances (with respect to different observers) from $O$ equal to this unit thus also form the rotational hyperboloid $c^2t^2 - x^2 - y^2 - z^2 = -1$) (rotating around the $t$-axis) with asymptotes on the light cone. Note that the lengths in directions perpendicular to the velocity are unaffected by the motion, while the $t$- and $x$- dimensions change in an inversely proportional ratio, so that the spacetime volume is invariant.

The relativistic transformation (the Lorentz's transformation) between inertial frames moving in the $x$-direction thus reads

$$\begin{aligned} ct' &= \gamma(ct - \beta x) , \\ x' &= \gamma(-\beta ct + x) , \\ y' &= y , \\ z' &= z . \end{aligned} \tag{2.5}$$

It is easy to see that the so-called spacetime interval[2]

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.6}$$

is invariant with respect to this transformation (this is also why the proper space $t' = 0$ of each observer intersects in all directions the hyperboloid $s^2 = +1$ of unit distances just in the unit sphere $x'^2 + y'^2 + z'^2 = 1$). The invariance of this indefinite quadratic form implies also invariance of the scalar product defined in the coordinates $t, x, y, z$ by the matrix (Minkowski tensor) $\eta_{\iota\kappa} = \mathrm{diag}(-c^2, +1, +1, +1)$. Lorentz transformations (in the same directions, boost) form a one-parametric group. Its parameter $\beta$ satisfies the rule of addition of velocities

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} , \tag{2.7}$$

as it can be seen multiplying matrices corresponding to (2.5)

$$\Lambda(\beta) \equiv \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} = \gamma_1\gamma_2 \begin{pmatrix} 1 & -\beta_1 \\ -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta_2 \\ -\beta_2 & 1 \end{pmatrix} . \tag{2.8}$$

**Exercise 1** *Explain the "car and garage paradox" (pp.9-10 of [6]): A car moves into garage of equal proper length and a doorman closes the door immediately as its back enters. According to him the Lorentz contracted car fits inside, while from the point of view of the driver the garage is too small. Who is right?*

*Discuss the variant "fast man on a net" of this paradox: The car is relativistically contracted with respect to a net with holes of the same proper length. According to the driver, the holes are contracted. Will the car fall down through the net or not?*

[2] Note that in the literature both signatures are used: $(-+++)$ e.g., in [4] or [6], $(+---)$ in [2] or [3].

**Exercise 2** *Phileas Fogg II. was asked to synchronize clocks in stations on the circum-equatorial railway. He used the method of light-ranging starting from the main station eastward. What was the result after coming back to the starting point?*

**Exercise 3** *Derive the relation (2.4) for the Lorentz $\gamma$-factor from the assumption that the so-called Bondi $K$-factor, i.e., the ratio of proper times of receiving the light signal by one observer to the time of sending it by the other ($\frac{t'_B}{t_{A-}} \equiv K \equiv \frac{t_{A+}}{t'_B}$ in Fig 2.2, right) is the same for both $S$ and $S'$. Derive the formula (2.7) using the $K$-factors for the transformations $S \to S'$, $S' \to S''$ and $S \to S''$ (investigate the light-ray $DBA_+$ in Fig. 2.2, right, from the point of view of each couple of systems).*

**Exercise 4** *Find the general formulae (according to special relativity) for the Doppler effect and aberration of light. Hint: Compare the components of the 4-momentum of a photon as seen by two observers moving with different velocities not aligned with the direction of the photon path.*

## 2.3 Accelerated observers

A non-inertial observer moving along the worldline $x^\iota = x^\iota(s)$ ($s$ being an arbitrary parameter), will measure on his (ideal) clock his proper time $\tau$ running at the same rate as the time of an inertial observer instantaneously comoving with him,

$$c\tau = \int \sqrt{-\eta_{\iota\kappa}dx^\iota dx^\kappa} \ . \tag{2.9}$$

He will thus measure the temporal variation of any scalar function $f(x^\iota)$ on the spacetime

$$\frac{d}{d\tau}f = \frac{dx^\iota}{d\tau}\frac{\partial}{\partial x^\iota}f(x) \ . \tag{2.10}$$

The differential operator $U \equiv \frac{d}{d\tau}$, named 4-velocity, is the vector tangent to the worldline (cf. Definition 7) and satisfying $(U, U) = -c^2$.

Let us investigate a one-dimensional accelerated motion $t = t(\tau)$, $x = x(\tau)$ of an observer who will measure acceleration $a = a(\tau)$ in his own space. After an infinitesimal increment $d\tau$ of his proper time, he will reach the velocity $dv' = a.d\tau$ with respect to the primed inertial observer comoving with him (with velocity $v$) at time $\tau$. With respect to a basic reference inertial observer he will thus reach the velocity

$$v + dv = \frac{v + dv'}{1 + v.dv'/c^2} = v + (1 - \frac{v^2}{c^2})dv' + o(v^2) \ . \tag{2.11}$$

Expressing $dv'$ from here and integrating, we arrive at

$$\int a d\tau = \int \frac{dv}{1 - \frac{v^2}{c^2}} = c.\mathrm{argtanh}(\frac{v}{c}) \ . \tag{2.12}$$

From here

$$v = c.\mathrm{tgh}\int \frac{a d\tau}{c} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\sqrt{1 - \frac{v^2}{c^2}} = \frac{dx}{d\tau}\frac{1}{\cosh\int a d\tau / c} \ , \tag{2.13}$$

and finally by integration

$$x = \frac{c^2}{a}\cosh\int \frac{a d\tau}{c} \tag{2.14}$$

and

$$t = \frac{c}{a}\sinh \int \frac{a d\tau}{c} \ . \tag{2.15}$$

It is possible to calculate from Eq. (2.13) also the coordinate acceleration observed by the static observer

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{d\tau}\frac{d\tau}{dt} = c\frac{a/c}{\cosh^2 \int a d\tau/c}\sqrt{1-\frac{v^2}{c^2}} = a\left(1-\frac{v^2}{c^2}\right)^{3/2} \ , \tag{2.16}$$

which is consistent with the special-relativistic dynamics according to which the inertial mass is a function of velocity

$$F \equiv m_0 a = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right) \ . \tag{2.17}$$

For a constant proper acceleration, the integration in Eqs. (2.14) and (2.15) reduces to a simple product $(a\tau/c)$. This result corresponds in the non-relativistic limit $c \to \infty$ to the standard result $t \simeq \tau$, $x \simeq x_0 + at^2/2$. The resulting worldline is (up to the scaling factor $a$ and arbitrary shift in $x$) identical with the hyperbola $AA'$ of events in unit distance from $O$ in Fig. 2.3 right. The lines $OA$, $OA'$ are thus proper spaces of the accelerated observer in different times. Worldlines in constant proper distances from this solution correspond to other solutions of (2.12) with initial (at $t = 0$) values of the acceleration $a = c^2/x$. The system of these worldlines (for the initial $x > 0$) fills only one quarter of the spacetime (above the past and below the future light cones of events $x = 0$, $t = 0$). Their parameters $\tau, a$ can be used as alternative curvilinear coordinates in this region of spacetime. These coordinates (proper times $\tau$ of the observers and $a$ or, alternatively, their proper distances $x' = c^2/a$ from $O$) are not orthogonal with respect to the Minkowskian metric. A set of time-orthogonal comoving coordinates $t', x'$ can be obtained introducing the 'coordinate' time $t' = a\tau/a_0$ (given by the proper time of one of the accelerated observers chosen as a reference one). The transform from the primed to the inertial coordinates reads

$$x = x'\cosh\frac{a_0 t'}{c} \ , \tag{2.18}$$

$$t = x'\sinh\frac{a_0 t'}{c} \ , \tag{2.19}$$

and the spacetime interval (in this so-called Rindler spacetime) reads

$$ds^2 = -c^2dt^2 + dx^2 = -\left(\frac{a_0 x'}{c^2}\right)^2 c^2dt'^2 + dx'^2 \ . \tag{2.20}$$

The proper time $\tau$ for each fixed $x'$ is thus running (compared to the coordinate time $t'$) at the rate proportional to $x'$, which can be interpreted by the accelerated observers as a consequence of the apparent gravitational acceleration in their noninertial system.

If a general dependence of gravitational acceleration on the distance (e.g., Newtonian one, $a = GM/x^2$) should be locally modelled by strips of the spacetime cut from Minkowski spacetime around worldlines of corresponding $a$ in this solution, then they can not be glued together in a flat spacetime (cf. exercise 8).

**Exercise 5** *Explain the twin paradox (calculate the proper times of world lines $OAE$ and $OBE$ in Fig. 2.2, right; investigate their instantaneous proper spaces).*

## 2.4 Electrodynamics

Electromagnetic field is in relativity described by a spacetime 2-form $F$ (i.e., by an antisymmetric tensor field) which acts on a charged particle with charge $q$ and rest mass $m_0$ (both are its scalar characteristics) and with 4-velocity $U = \frac{d}{d\tau}$ ($\tau$ being the proper time of the particle) by the Lorentz force resulting in its 4-acceleration

$$a^\iota \equiv \frac{D}{d\tau}U^\iota = \frac{q}{cm_0}F^\iota{}_\kappa U^\kappa \ . \tag{2.21}$$

This dynamical law is obviously consistent with the normalization condition $U^2 = -c^2$, because $\frac{d}{d\tau}U^2 = 2(Ua) = 0$ due to the antisymmetry of $F$.

It corresponds also to the non-relativistic view where in a chosen observers reference frame there is $F$ described by two 3-vectors $E$ of an electric and $B$ of a magnetic field[3]

$$F_{\iota\kappa} = \begin{array}{cc} & \begin{matrix} \kappa=0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} \iota=0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix} \end{array} \ . \tag{2.22}$$

In this "3+1"-formalism we then get for components $U^\iota = \gamma(c, v^i)$ linearly dependent equations

$$\frac{d(\gamma v^i)}{d\tau} = \frac{\gamma q}{m_0}(E^i + \epsilon^{ikl}\frac{v_k}{c}B_l) \ , \tag{2.23}$$

$$\frac{d\gamma}{d\tau} = \frac{\gamma q}{cm_0}E^k v_k \ . \tag{2.24}$$

The components of $F$ are interrelated between different reference frames by a usual Lorentz transformation. For a pure space rotation both $E$ and $B$ are transformed like vectors (because they are defined by the same $u$). On the other hand, for a pure boost in direction $n$ (cf. Eq (2.8))

$$\Lambda^\iota{}_\kappa = \begin{pmatrix} \gamma & \gamma\beta n \\ \gamma\beta n & \mathbf{1} + (\gamma - 1)n \otimes n \end{pmatrix} \ , \tag{2.25}$$

they are mixed by the mutual induction

$$E' = [\gamma\mathbf{1} + (1-\gamma)n \otimes n]E + \gamma\beta n \times B \ , \tag{2.26}$$

$$B' = [\gamma\mathbf{1} + (1-\gamma)n \otimes n]B - \gamma\beta n \times E \ , \tag{2.27}$$

(where $\times$ is the vector product – cf. Definition 4). There can be found reference frames in which $E$ and $B$ are parallel, and

$$F = -e\, dx^0 \wedge dx^1 + b\, dx^2 \wedge dx^3 \ . \tag{2.28}$$

[3] These can be defined for a given 4-velocity $u$ of the observer as 4-vectors perpendicular to it by

$$E^\iota \equiv \frac{1}{c}F^\iota{}_\kappa u^\kappa \ ,$$

$$B_\iota \equiv \frac{1}{2c}\epsilon_{\iota\kappa\lambda\mu}u^\kappa F^{\lambda\mu} \ .$$

In different systems of units these vectors are also measured by independent units.

There exist two scalar characteristics of $F$ (which are thus Lorentz invariant)

$$|F|^2 = F^{\iota\kappa} F_{|\iota\kappa|} = B^2 - E^2 \ , \tag{2.29}$$

and

$$\frac{1}{4} \langle\, {}^*\!F, F \rangle = \epsilon^{\iota\kappa\lambda\mu} F_{|\iota\kappa|} F_{|\lambda\mu|} = (EB) \ , \tag{2.30}$$

where the dual 2-form ${}^*\!F$ has components

$${}^*\!F_{\iota\kappa} = \epsilon_{\iota\kappa\lambda\mu} F^{\lambda\mu} = \left( \begin{array}{c|cccc} & \kappa=0 & 1 & 2 & 3 \\ \hline \iota=0 & 0 & B^1 & B^2 & B^3 \\ 1 & -B^1 & 0 & E^3 & -E^2 \\ 2 & -B^2 & -E^3 & 0 & E^1 \\ 3 & -B^3 & E^2 & -E^1 & 0 \end{array} \right) \ . \tag{2.31}$$

The dynamical law for $F$ are the Maxwell's equations which can be written using the $n$-forms formalism in a compact form

$$dF = 0 \ , \tag{2.32}$$

$$\delta F \equiv {}^*d^*F = 4\pi j \ , \tag{2.33}$$

where $j$ is the 4-current density and $\delta \equiv {}^*d^*$ is the 'interior derivative'. The first of these equations can be satisfied if the 4-potential $A$ is introduced so that

$$F = dA \ . \tag{2.34}$$

The 4-potential is determined by this equation up to the gauge transformation $A \to A'$,

$$A' = A + df \ , \tag{2.35}$$

where $f$ is an arbitrary scalar function. Maxwell's equations (2.32) and (2.33) can be written in components as

$$F_{\iota\kappa,\lambda} + F_{\kappa\lambda,\iota} + F_{\lambda\iota,\kappa} = 0 \ , \tag{2.36}$$

$$F^{\iota\kappa}{}_{,\kappa} = 4\pi j^\iota \ , \tag{2.37}$$

and in "3+1"-formalism

$$(\vec{\nabla}.\vec{B}) = 0 \ , \qquad \partial_0 \vec{B} + \vec{\nabla} \times \vec{E} = 0 \ , \tag{2.38}$$

$$(\vec{\nabla}.\vec{E}) = 4\pi\rho \ , \qquad \partial_0 \vec{E} - \vec{\nabla} \times \vec{B} = 4\pi\vec{j} \ , \tag{2.39}$$

where $\rho \equiv j^0$ is the charge density. Eq. (2.34) reads in these formalisms

$$F_{\iota\kappa} = A_{\iota,\kappa} - A_{\kappa,\iota} \ , \tag{2.40}$$

$$\vec{B} = -\vec{\nabla} \times \vec{A} \ , \qquad \vec{E} = \partial_0 \vec{A} - \vec{\nabla}\varphi \ . \tag{2.41}$$

These equations can be obtained from a variational principle with the Lagrange density of free electromagnetic field

$$\mathcal{L} = -\frac{1}{16\pi c} F^{\iota\kappa} F_{\iota\kappa} \ , \tag{2.42}$$

where the variable is the 4-potential $A$, so that (2.36) is implicitly satisfied and the explicit form of the Lagrangian is

$$\mathcal{L} = -\frac{1}{8\pi c} \left( A^{\iota,\kappa} A_{\iota,\kappa} - A^{\kappa,\iota} A_{\iota,\kappa} \right) ; . \tag{2.43}$$

According to (B.22) there follows (2.37).

The stress-energy tensor can be chosen symmetric as

$$T_\iota{}^\kappa = \frac{\partial \mathcal{L}}{\partial A_{\lambda,\kappa}} A_{\lambda,\iota} - \delta_\iota^\kappa \mathcal{L} + \frac{1}{4\pi c} F^{\lambda\kappa} A_{\iota,\lambda} \ , \tag{2.44}$$

i.e.,

$$T_\iota{}^\kappa = \frac{1}{4\pi c} \left( -F^{\lambda\kappa} F_{\lambda\iota} + \frac{1}{4} \delta_\iota^\kappa F^{\lambda\mu} F_{\lambda\mu} \right) \ . \tag{2.45}$$

In "3+1" formalism

$$4\pi c T^{\iota\kappa} = \left( \begin{array}{c|cccc} & \kappa=0 & 1 & 2 & 3 \\ \hline \iota=0 & -\frac{E^2+B^2}{2} & & E \times B & \\ 1 & & & & \\ 2 & E \times B & & E \otimes E + B \otimes B - \mathbf{1}\frac{E^2+B^2}{2} & \\ 3 & & & & \end{array} \right) \ . \tag{2.46}$$

The interaction Lagrangian is $\frac{q}{c}(Aj)$.