1.3 Newtonian cosmology

The expansion of the Universe and some of its properties can be understood in the framework of non-relativistic mechanics – the so-called *Newtonian cosmology.* On the surface of a homogeneous sphere with radius r and density ρ (and, consequently, the mass $M = \frac{4}{3}\pi r^3 \rho$) is acting the gravity acceleration

$$g = \frac{GM}{r^2} = \frac{4}{3}\pi G\rho r \tag{9}$$

towards the centre of the sphere. Any thought sphere of radius r_0 in an infinite homogeneous medium

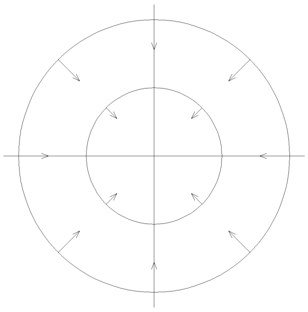

Figure 3: Gravitational collapse of infinite homogeneous medium.

(which is initially at time t_0 in rest and which is not held by any repulsive force) must thus collapse ($r = r(t, r_0)$) with the acceleration of the surface $\ddot{r} = -g \sim r_0$. Consequently, also the velocity $v \equiv \dot{r}$ must be proportional to r_0 in any time, i.e. it satisfies the Hubble's relation

$$v = Hr\ , \tag{10}$$

where the so-called *Hubble constant* H is independent on the distance r, but generally it is a function of time. Obviously, the relative motion of any two elements of the medium is independent on the choice of the center of the spheres.

Because the mass of the sphere is constant, its density is increasing with the time, $\rho = \rho(t) \sim r^{-3}(t)$, but it remains homogeneous. The kinetic energy per unit mass of the relative motion of the surface with respect to the centre will be

$$E_{\mathrm{kin}} = \frac{1}{2}\dot{r}^2 = E - V = E + \frac{4}{3}\pi G\rho r^2\ , \tag{11}$$

where E is the total energy which is constant and $V = -GM/r$ is the potential energy, which is getting more and more negative in the course of the collapse. It is obvious from the analogy with the vertical motion in a gravitational field that for $E < 0$, there must precede an expansion (from $r = 0$ to $r = r_{\mathrm{max}}$) to the contraction (back to $r = 0$ in finite time), and for $E \geq 0$ there are two possible solutions: a contraction from $r = \infty$ at $t = -\infty$ to $r = 0$ in finite t, or an expansion from $r = 0$ in finite t to $r = \infty$ in $t = \infty$ – cf. Tab. 1 and Fig. 4. The critical case $E = 0$ gives a boundary between the solutions of infinite expansion (or contraction from infinity) and the expansion deccelerated and turned back into a collapse. These types of dynamics can be distinguished by comparing the relation between the instantaneous value of Hubble constant and the density. Alternatively, the density can be determined by the rate of the gravitational deceleration, which is usually expressed in terms of the dimension-less so-called *deceleration parameter*

$$q = -\frac{\ddot{r}r}{\dot{r}^2}\ , \tag{12}$$

Comparing equations (11) and (4) for $\Lambda = 0$, one can see that the radius r of any sphere in the Newtonian cosmology satisfies the same equation of energy conservation as the radius R of the

Universe in the Friedmann model where the term $-\frac{1}{2}c^2k$ plays the role of the energy E. In the case of dust with $P = 0$ (or at least $P \ll \rho c^2$), the Newtonian cosmology gives the same dynamics of the radius $R = R(t)$ of the Universe as the Friedmann models.

Obviously, for $r(t) \to 0$ the potential energy of the sphere $V \to -\infty$ and the constant energy of the motion is neglegible ($E \ll |V|$) on the right-hand side of Eq. (11), and consequently $\dot{r} \sim r^{-1/2}$ and $r^{3/2} \sim t$. This is valid precisely for all r in the critical case, corresponding to the relativistic case with $k = 0$, $P = 0$ and $\Lambda = 0$.

Exercise 3 *Find the relation between the type of the dynamics of the Newtonian model and the instantaneous values of H and ρ or q.*

Exercise 4 *Find the age of Universe in terms of the instantaneous value of H for the critical case ($E = 0$) and for the extreme case of zero density.*