1.2 Relativistic cosmology

A qualitatively new insight into cosmology has been opened by general relativity which replaced the study of distribution and motion of matter in a fixed Euclidean space by a non-Euclidean geometry of the space-time. The matter (either particles or fields) moves in a curved space-time like in a flat Minkowskian space-time which is its locally tangent space (e.g. free particles move on geodetics) but at the same time the stress-energy tenzor of the matter produces the curvature of the space-time according to the Einstein field equations[9]

$$\mathcal{R}_{\iota\kappa} - \frac{1}{2} g_{\iota\kappa}\mathcal{R} - \Lambda g_{\iota\kappa} = 8\pi G T_{\iota\kappa} \; . \tag{1}$$

This so-called geometrodynamics thus mutually causally connects the space-time geometry with the distribution and motion of the mass and enables thus the cosmology to study the whole space of the Universe as an attribute of the matter.[10] A.A. Friedmann and W. Robertson showed that following the cosmological principle, the Universe must have in a fixed time the geometry of space with constant curvature (the so-called *Friedmann – Robertson – Walker models* of the Universe).[11] The curvature of the space can be either positive or zero or negative, and can be characterized by the index of curvature $k = +1, 0, -1$, accordingly. The case $k = 0$ corresponds to the Euclidean geometry of flat space (the circumference of any circle with radius r reads $o = 2\pi r$; the so called *Einstein – de Sitter model*). The case $k = -1$ corresponds also to an infinite (open) universe, however, the geometry of the space is that of a saddle surface ($o > 2\pi r$). Finally, in the case $k = +1$ the space has the geometry of a supersphere ($o < 2\pi r$) and has a finite volume (it is closed; both cases $k = \pm 1$ are called *Friedmann – Lemaître models*) – see Fig. 2. If we introduce generalized spherical coordinates

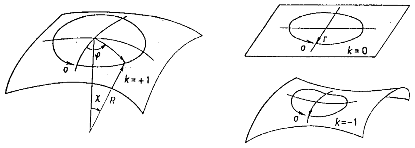

Figure 2: The geometry of homogeneous isotropic spaces.

$\{\chi, \vartheta, \varphi\}$ in the space, the metric of the space-time will be given by the space-time interval[12]

$$ds^2 = c^2 dt^2 - R^2(t)[d\chi^2 + \Sigma^2(\chi)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)] \; , \tag{2}$$

where

$$\Sigma(\chi) = \left\langle \begin{array}{l} \sin\chi \\ \chi \\ \sinh\chi \end{array} \right. , \quad \text{for} \quad k = \left\langle \begin{array}{l} +1 \\ 0 \\ -1 \end{array} \right. , \tag{3}$$

and R is a time-dependent radius of curvature of space (or a scaling factor in the case $k = 0$). From the Einstein field equations, there follow equations of motion for the geometry of space (characterized by R) and for the density of mass

$$\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} - \Lambda\frac{c^2}{3} = \frac{8\pi G}{3}\rho \; , \tag{4}$$

[9] In units $c = 1$, $\mathcal{R}_{\iota\kappa}$ are the components of Ricci tenzor – for details see the lectures on general relativity.

[10] In this respect the general relativity satisfies the *Mach's principle*, according to which the inertial systems are priviledged owing to their non-accelerated motion with respect to the distribution of mass in the Universe.

[11] It means that the four-dimensional space-time, which according to the special relativity is locally invariant to the Lorentz transformation, can be sliced by a special choice of the proper time t of cosmic matter into a set of three-dimensional spaces, which are homogeneous and isotropic, and their geometry has the same curvature everywhere in space and is dependent on t only.

[12] The same space-time geometry is used to be alternatively expressed also in other coordinate systems – cf. exercise 1.

$$\frac{2\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}+\frac{kc^2}{R^2}-\Lambda c^2=-\frac{8\pi G}{c^2}P\ , \tag{5}$$

where G is the gravitational constant and c is the speed of light. The components of the stress-energy tensor of the matter in the universe, the mass density ρ, and the pressure P, are functions of time t only, and there follows for them from Eqs. (4) and (5) the equation of continuity of mass-energy,[13]

$$\frac{d}{dt}(\rho c^2 R^3)+P\frac{dR^3}{dt}=0\ . \tag{6}$$

The first term can be interpreted as the change in unit time of the total energy of the mass contained in the space, while the second one is the work exerted by the pressure due to the change of the volume of the Universe. The term proportional to the *cosmological constant* Λ was introduced by Einstein into his gravitational law to enable the existence of a stationary solution ($\dot{R}=0$) of equations (4) and (5). The value of the cosmological constant $\Lambda>0$ represents, in fact, a repulsive force which could hold the Universe apart against the attractive gravitational force. However, just the Friedmann dynamical models enable us to explain the cosmological paradoxes and they give several theoretical predictions, which are confirmed by observations.

Exercise 1 *Find the form of Eq. (2) in the case that the radial space coordinate χ is replaced by another coordinate ρ by substitution $\chi=\chi(\rho)$. Find the special cases of this substitution and the corresponding resulting metrics which lead to*

$$ds^2=c^2dt^2-R^2(t)[f(\rho)d\rho^2+\rho^2(d\vartheta^2+\sin^2\vartheta d\varphi^2)]\ , \tag{7}$$

or to the conformly flat coordinates

$$ds^2=c^2dt^2-R^2(t)f(\rho)[d\rho^2+\rho^2(d\vartheta^2+\sin^2\vartheta d\varphi^2)]\ . \tag{8}$$

Exercise 2 *Prove that Eq. (6) follows from Eqs. (4) and (5).*

[13] This equation is a special case of the general equation of continuity, $T^{\iota\kappa}{}_{;\kappa}$, which follows from the Einstein field equations (1).